EUROPEAN SCIENCE FOUNDATION
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RESEARCH CONFERENCES

# Symmetries and Integrability of Difference Equations – SIDE 9

Varna ▪ Bulgaria
14-18 June 2010

http://old.inrne.bas.bg/SIDE-9/

# SCIENTIFIC REPORT

# Conference Highlights

*Please provide a brief summary of the conference and its highlights in non-specialist terms (especially for highly technical subjects) for communication and publicity purposes. (ca. 400-500 words)*

The subject of SIDE-9 conference: discrete and difference equations stands at the frontier of pure and applied mathematics, and physics as well as in the applied mathematical sciences such as theoretical biology or mathematical finance. The theory of discrete equations is still developing, and the integrability aspects, which are the center of interest of the meeting are the best line of approach: it is an extremely rich field, related to algebra, analysis, complexity theory, "quantum geometry" (and quantum physics), algebraic geometry, and even number theory. SIDE 9 is the ninth in a series of biennial conferences devoted to these topics, which include also ordinary and partial difference equations, analytic difference equations, orthogonal polynomials and special functions, symmetries and reductions, difference geometry, integrable discrete systems on graphs, integrable dynamical mappings, discrete Painlevé equations, singularity confinement, algebraic entropy, complexity and growth of multivalued mapping, representations of affine Weyl groups, quantum mappings and quantum field theory on the space-time lattice.

Integrable systems are important for several reasons:
i) for their intrinsic mathematical and physical interest, e.g. as universal objects of dynamics;
ii) because their study has generated novel mathematical theory and computational techniques;
iii) as a starting point for a perturbative analysis of classes of non-integrable equations;
iv) as tests for analytical and numerical methods.
It is one of the miraculous facts established over the last two decades that practically all continuous integrable systems have exact discrete analogues described by *difference* rather than differential equations and sharing many properties with their continuous counterparts. Such discrete integrable systems (DIS) can be considered to be even more fundamental, since systematic expansions in the lattice grid parameter allows one to derive from a single discrete integrable equation an infinite number of integrable continuous equations. Therefore the study of DIS allows important inroads into new mathematical territories.

Furthermore, rendering dependent variables to be discrete (ultradiscrete systems), new rich fields have been found such as integrable cellular automata, for which a new calculus related to tropical geometry and valuation fields is in development. Discrete integrable systems play an important role also on the quantum level, as well in connection with conformal field theory and via the Baxter equation for integrable spin systems, and in a recent development of *quantum geometry*. They enter in the novel field of *discrete differential geometry* (i.e. the discrete counterpart of classical differential geometry of curves and surfaces) through connections with Darboux and Bäcklund transforms, which is a rich area for future applications, e.g. in visualisation problems and mathematical design. All these facts show, that the investigation of DIS may and is bringing new fruitful results in other fields of contemporary mathematics and theoretical physics.

☒ I hereby authorize ESF – and the conference partners to use the information contained in the above section on 'Conference Highlights' in their communication on the scheme.

# Scientific Report

## Executive Summary

*(2 pages max)*

The Conference SIDE 9 took place at the International House of Scientists "Frederic Joliot-Curie" (IHS) on the Black Sea coast near Varna, Bulgaria, from 14 to 18 June 2010. Varna is Bulgaria's third largest city and main naval port. The International House of Scientists is situated in the big international sea-side resort "St. Constantine and Elena" to the North-East of Varna.
The conference was organized by
**- the Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences**
**- the Institute of Informatics and Mathematics, Bulgarian Academy of Sciences.**

It was sponsored and supported by

- European Science Foundation
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- European Mathematical Society
- European Physical Society
- Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences

Conference webpage is http://old.inrne.bas.bg/SIDE-9/

The conference was attended by 59 participants from scientific institutions from all over the world. More than 30% of the participants were young people – PhD students and young fellows on post doc positions. The list of participants, the conference program with the titles and abstracts of the talks, and most of the presentations are on the webpage of the conference.

Arrival day of the conference was 13.06.2010; departure day was 19.06.2010

The conference started on Monday, 14.06.2010, 9:00h. 45 talks and 4 posters were presented at the conference.

## Scientific Content of the Conference

*(1 page min.)*

- *Summary of the conference sessions focusing on the scientific highlights*
- *Assessment of the results and their potential impact on future research or applications*

The conference was structured into the following conference sessions:

**"Discrete Painlevé & Garnier Equations and Nonlinear Special Functions"**
There were several talks devoted to the Painleve equations. Joshi's talk was on the utilization of geometry to find and analyze properties of solutions of discrete Painleve equations. Goryuchkina discussed the formal solutions to the Painleve equations represented in asymptotic expansions of five types: the power, the power-logarithmic, the complicated, the exotic and the half-exotic forms. All expansions of the five types near three singular and non-singular points of the sixth Painleve equation for all values of its four complex parameters are found. K. Kajiwara analyzed the ultradiscretization of solvable chaotic system associated with Hesse cubic curve.

In his talk J. Hietarinta proposed that the existence of three-soliton solutions can in practice be viewed as a criterium for integrability. This provides a method for searching for integrable equations. A. Ramani presented an extension of the QRT mapping beyond the familiar symmetric and asymmetric varieties. Using our results on discrete Painleve equations, integrable QRT-like mappings were found whose coefficients are periodic functions. It was shown that there exist cases, where the periods are arbitrarily long. The integrability of all the examples is proven and it is shown how they can be explicitly integrated in terms of elliptic functions. The last report in this session by Y. Shi was on q-discrete Painleve equation and its associated linear problem.

**"Characterisations of integrability"**

Five talks by G. Grahovski, V. Gerdjikov, V. Dryuma, M. Visinescu and A. Visinescu were devoted to the analysis of integrable nonlinear evolution equations. G. Grahovski studied the soliton perturbations of the Camassa-Holm hierarchy. He used the so-called generalised Fourier transform (GFT) which provides a natural setting for the analysis of small perturbations to an integrable equation: starting from a purely soliton solution one can 'modify' the soliton parameters such as to incorporate the changes caused by the perturbation. V. Gerdjikov proposed a generalization of the Ablowitz-Ladik method to Lie groups of higher rank. As a result one can derive hierarchies of integrable difference evolution equations which in the continuous limit go into the N-wave equations, multicomponent nonlinear Schrodinger equations and others. V. Dryuma analyzed multidimensional Plebanski metrics and their properties. He constructed particular solutions of nonlinear differential equations defined by the Ricci-flat 6D and 8D Plebanski metrics and studied the properties of the associated Beltrami-Laplace operators. A. Visinescu reported on periodic and solitary wave solutions of two-component Zakharov-Yajima-Oikawa system. Using Madelung's approach she found the velocities of the periodic solutions, expressed through Jacobi elliptic functions. M. Visinescu talked about higher order symmetries and quantum gravitational anomalies in a covariant Hamiltonian formulation. He showed that the quantum gravitational anomalies are absent if the hidden symmetries are associated with Killing-Yano tensors. However conformal Killing tensors are sources of gravitational anomalies even if they are square of conformal Killing-Yano tensors.

**"Discrete Differential Geometry"**

A. Doliwa reported on the affine Weyl group symmetry of Desargues maps and quadrilateral lattices. Decio Levi presented generalized symmetry integrability test of discrete equations on the square lattice. Imposing the existence of a generalized symmetry for equations defined on a square lattice he obtained an integrability test which he used to classify some nonlinear affine equations which pass the multiple scale A3 test.

M. Nieszporski reviewed results on integrable discretization of Weingarten rectilinear congruences.

R. Rebelo introduced definitions of finite differences approximating partial derivatives on potentially non orthogonal lattices are given and used them as new variables in the discrete space. Thus he obtained symmetry preserving discretization of PDEs.

**“Quantum Discrete Systems and Quantum Geometry”**

The session contained 7 talks covering different aspects. The talk by S. Sergeev “Quantum Q4” presented an important breakthrough by establishing a master solution of the star-triangle relation with Boltzmann weights defined through Ruijsenaars' elliptic Gamma-function (recent joint work with V. Bazhanov). In a quasi-classical limit a connection with Adler's Q4 lattice equation was established. The contributions of V. Papageorgiou and T. Kouloukas dealt with the Yang-Baxter maps, in particular the ones associated with the Adler-Yamilov discretization of Landau-Lifschitz equations. S. Lobb, P. Xenitidis and Yu. Suris discussed the recent progress in the Lagrangian formulation of multidimensionally consistent lattice equations, in particular those of the ABS list. The important property here is the “closure relation” (also called “flip invariance”) of Lagrangians on the multidimensional lattice, which provides an important novel interpretation of the Lagrangians in terms of multiforms. Finally, the talk by M. Yakimov dealt with

representation-theoretical aspects of quantum groups, in particular it provided a characterization of the spectra of quantum partial flag varieties and quantum Schubert cells.

**Special session "Cluster algebras and Poisson maps"**
There were three talks in this session. The first, by Fordy, introduced a combinatorial construction of recurrence relations from certain quivers (directed graphs), and explained the classification (by Fordy and Marsh) of all quivers having a periodicity property under cluster mutation. The associated recurrences in some cases correspond to maps that are completely integrable (even superintegrable) in the Liouville sense, and also linearizable. The second talk, by Hone, continued the talk of Fordy by describing how to use the quiver to construct a symplectic form, in general on a space of lower dimension, so that the corresponding map becomes symplectic. In particular, this construction yields a Poisson structure for travelling wave-type reductions of the Hirota-Miwa equation, a well known integrable partial difference equation, and for other reductions such as Lyness maps. (The latter maps were discussed in the talk by Dinh Tran, in a different session.) The third and final talk in this session was by Sebastian Zwicknagl, who discussed Poisson ideals in cluster algebras. He started from a more general type of quiver than those treated by Fordy, and explained how such a quiver leads to a cluster algebra admitting a family of Poisson structures that are compatible with the maps generated by cluster mutation. He then presented new results on the explicit construction of symplectic foliations, whereby the cluster algebra is viewed as the coordinate ring of an affine variety, which is stratified into finitely many torus orbits of symplectic leaves. Overall the session showed that there are fruitful connections between cluster algebras and discrete integrable systems, which should lead to a deeper understanding of the latter in terms of combinatorics, algebraic and Poisson geometry.

**"Symmetries and conservation laws of Discrete Equations"**
Altogether 10 contributions were presented in this session.
Among the most relevant ones let us mention the algebraic construction of conditions for the existence of a hierarchy of higher symmetries and its application to partial difference equations on a square by Levi, Mikhailov and Wang. They have been able to obtain an infinite set of conditions necessary for the existence of an infinite set of generalized symmetries related by a recurrence relation. Already the first few of these conditions provide sufficient conditions for the integrability of classes of nonlinear partial difference equations on the lattice. C. Scimiterna provided a set of integrability conditions for a class of discrete systems defined on the square. The presentations by Kozlov, Nteumagne, Viallet and Ormerod showed the importance of the notions of invariance, first integrals and symmetries in the description of discrete systems on regular or exponential lattices. The infinitesimal formalism for the description of the symmetries or lambda symmetries of difference or differential difference equations has been clarified in the presentations of Levi and Winternitz showing that in many cases the relation between the continuous case and the discrete is quite subtle. It is often necessary to go back to he basic formulation of the involved problems and one cannot just intuitively discretize the continuous case. The 10 presentations provided new results in a rapidly developing area of fundamental research.

**"Orthogonal Polynomials and Q-difference Equations"**
L. Boelen analyzed orthogonal polynomials and discrete Painleve equations by noticing that the recurrence coefficients of certain semiclassical orthogonal polynomials solve discrete Painleve equations. She found solutions that can be computed uniquely using the boundary conditions, which arise naturally in the orthogonal polynomial context. P. Spicer talked on higher analogues of the discrete-time Toda equation and the quotient-difference algorithm. The discrete-time Toda equation arises as a universal equation for the the relevant Hankel determinants associated with one-variable orthogonal polynomials through the mechanism of adjacency, which amounts to the inclusion of shifted weight functions in the orthogonality condition. This mechanism is extended to a new class of two-variable orthogonal polynomials where the variables are related via an elliptic curve. This leads to an 11-point quadrilinear equation for the associated Hankel determinants, together with its Lax pair, which is derived from the relevant recurrence relations for the orthogonal

polynomials. In a similar way as the quotient-difference (QD) algorithm is related to the discrete-time Toda equation, a novel quotient-quotient-difference (QQD) scheme is presented for the 11-point quadrilinear equation. R. Sasaki's talk was devoted to exceptional orthogonal polynomials and difference Schrodinger equations. Recently, several series of infinitely many exceptional orthogonal (Laguerre, Jacobi, Wilson and Askey-Wilson) polynomials were constructed by Odake-Sasaki as exact solutions of the ordinary and difference Schrodinger equation in one dimension. These polynomials are exceptional in the sense that they start from degree L>1 and thus not constrained by any generalisation of Bochner's theorem.

**"Integrability Detectors"**

The topic of this session was analyzed in the reports by J. Satsuma**,** V. Papageorgiou, and C. Viallet. Satsuma's report presented several ultradiscrete analogues of nonlinear integrable systems and discussed on the properties of their solutions. The various types of Yang-Baxter (YB) maps were explained by V. Papageorgiou. He started with the lattice equations such as (KdV, Bousinesque) and and related to them degenerate YB maps. Next he explained how one can go from degenerate to non-degenerate YB maps. Finally he presented YB maps for Lax operators on elliptic curves. Viallet's talk was devoted to the invariants of discrete integrable systems. As a criterion of integrability he proposes the vanishing of the algebraic entropy. Then he shows that the degree of the map grows polynomially rather that exponentially. If the growth is linear the linearizability follows. Quadratic growth is standard for "classical" models like QRT or discrete Painleve. Higher rates of growth were also analyzed.

**"Random Matrix Theory and Isomonodromy Transformations"**

As for the scientific highlights of the conference interesting results were the work by Levi, Mikhailov and Wang on existence of higher order symmetries of partial difference equations, and also the results presented by Lobb, Xenitidis and Suris on Lagrangian structures of lattice equations. P. Iliev's talk was on bispectrality of multivariable orthogonal polynomials**.** The polynomials orthogonal on the real line satisfy a three term recurrence relation, i.e. they are eigenfunctions of a second-order difference operator acting on n. The polynomials which are also eigenfunctions of a second-order differential or difference operator acting on x have many applications and can be obtained as reductions or limits of the Askey-Wilson polynomials. Some classification results and bispectral properties for multivariable orthogonal polynomials are given.
O. Kounchev discussed a new new point of view for the multivariate orthogonality and isospectral deformations. By introducing the so-called pseudopositive measures a new approach to multivariate orthogonality is proposed, which can be a source of new integrable models, and new isospectral deformations. A. Dzhamay proposed geometric configurations related to matrix factorizations. He studied rational Lax matrices represented as products of Blaschke-Potapov factors and showed that the exchange rules for the components of such factors can be visualized by labeled atomic trivalent graphs. In particular, the configuration corresponding to the transposition of two such blocks is a cube and both the correct choice of the Lagrangian coordinates and the expression for the Lagrangian itself can be readily obtained from the labeling of the edges and the vertices of this cube.

## Forward Look

*(1 page min.)*

- *Assessment of the results*
- *Contribution to the future direction of the field – identification of issues in the 5-10 years & timeframe*
- *Identification of emerging topics*

A number of important open problems were pointed out during the conference which will attract the attention not only of the participants, but also of their colleagues around the world. We briefly mention only two of them.

We start with the problem of finding suitable i*ntegrability detectors* for discrete systems and nonlinear difference equations, where the algebraic geometry of rational surfaces, notions of complexity in terms of algebraic entropy, the theory of growth of meromorphic functions (Nevanlinna theory), and Diophantine approximations in number theory have been used as sources of discrete integrability detectors.

Another important topic is the quantum discrete systems and quantum geometry. The possibility to obtain

a master solution of the star-triangle relations with Boltzmann weights is important also for applications in statistical physics.

SIDE belongs to a series of biennial conferences. The next two conference sites have been officially accepted by the steering committee. SIDE-10 take place in China in 2012 and SIDE-11 will take place in India in 2014.

- Is there a need for a foresight-type initiative?

## Atmosphere and Infrastructure

- *The reaction of the participants to the location and the organization, including networking, and any other relevant comments*

A number of discussions friendly an collegial atmosphere took place not only during the sessions of the conference, but also during the coffee breaks and the free time. It was enhanced by the conference dinner and by the sightseeing of the center of the city of Varna which was conducted in the free afternoon. The participants were showing satisfaction with both the location and the organization of the conference.

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