Minimal types in stable Banach spaces

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(largely joint with Saharon Shelah)

Henson's Conjecture

- **Henson's Conjecture**: Every uncountably categorical elementary class of Banach spaces is "very close" to the class of Hilbert spaces.
- A class is called *elementary* if it is closed under ultra-products and ultra-roots (of Banach spaces).
- A class K is *uncountably categorical* if for some uncountable λ, every $B_1, B_2 \in K$ of density character λ are isometric.
- Equivalently, a class is elementary if and only if it is axiomatizable in an appropriate logic (e.g. Henson's logic, continuous logic).

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Motivations from Banach space theory

- **Theorem** (Krivine, Maurey) Every stable Banach space contains an almost isometric copy of an ℓ_p space for some $p \geq 1$.
- This addresses an old problem of Banach.
- Natural question: which Banach spaces are stable? Which elementary classes of Banach spaces are stable?
- An easier question: which elementary classes of Banach spaces are uncountably categorical? A typical example: the class of Hilbert spaces (there are other examples).

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Quantifier free types over Banach spaces

- A 1-quantifier free (complete) type over a Banach space B is a collection of statements of the form $\{\|a+x\| = r_a : a \in B\}$ such that for every $a_1, \ldots, a_k \in B$ and $\varepsilon > 0$ there is $b \in B$ such that

$$\left| \|a_i + b\| - r_{a_i} \right| \leq \varepsilon$$

- Equivalently, 1-quantifier free type p over a Banach space B is a collection of statements of the form $\{\|a+x\| = r_a : a \in B\}$ such that there is an ultrapower $\mathcal{B}$ of B and $b \in \mathcal{B}$ such that $\|a+b\| = r_a$ for all $a \in B$.
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- A *partial* type over B is a subset of a complete type over B.

The setting

- From now on, let K be an elementary class of Banach spaces (possibly with extra-structure).
- Let $\mathcal{B}$ be a very saturated model in K.
- When the word "formula" $\varphi(\bar{x}, \bar{a})$ is mentioned, it is OK to assume that

$$\varphi(\bar{x}, \bar{a}) = \left\| \sum_{i<k} \lambda_i a_i + \sum_{i<m} \sigma_i x_i \right\|$$

- When the word "type" is mentioned, it is OK to assume that we mean a q.f. 1-type in the language of Banach spaces.

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Wide types

- *Definition.* We call a partial type in 1 variable $\pi(x)$ (possibly with parameters) *wide* if the set of realizations of $\pi(x)$ in $\mathcal{B}$ contains the unit sphere of an infinite dimensional subspace of $\mathcal{B}$.
- The type $x = x$ is wide.
- *Theorem - Existence of Wide Types*(Dvoretzky). Let $\pi(x)$ be a wide partial type, A a set containing the domain of π, Δ a collection of formulae closed under connectives. Then there exists a complete Δ-type p over A containing π which is wide.

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Nonforking wide types

- *Theorem - Existence of Nonforking Wide Types.* Let $p(x)$ be a wide complete type over a model M, A a set containing M, Δ a collection of formulae closed under connectives. Then there exists a complete Δ-type p' over A containing p which is does not fork over M and is wide.

Dvoretzky Theorem

- Let B be a Banach space, $S(B)$ the unit sphere of B, $f \colon S(B) \to \mathbb{R}$. The *spectrum* $\gamma(f)$ is the collection of all $r \in \mathbb{R}$ such that for every $\varepsilon > 0$ and any integer k there exists a k-dimensional subspace F of B such that $|f(x) - r| \leq \varepsilon$ for all x in the unit square of F.
- Let B, f be as before. We denote by $\gamma'(f)$ the collection of all $r \in \mathbb{R}$ such that for any k and ε as above, F can be chosen to be $(1+\varepsilon)$-isomorphic to a k-dimensional Hilbert space.
- *(Dvoretzky-Mil'man Theorem)* Let f be a uniformly continuous function on the unit sphere of an infinite dimensional Banach space B. Then the spectrum $\gamma(f)$ is not empty. Moreover, $\gamma'(f)$ is not empty.

Stability of Banach Spaces

- *Definition* (Krivine, Maurey, 1981): A Banach space B is called *stable* if for every two bounded sequences $\langle a_n : n < \omega \rangle$ and $\langle b_n : n < \omega \rangle$ in B and an ultrafilter $\mathfrak{U}$ on ω,

$$\lim_{n,\mathfrak{U}} \lim_{m,\mathfrak{U}} \|a_n + b_m\| = \lim_{m,\mathfrak{U}} \lim_{n,\mathfrak{U}} \|a_n + b_m\|$$

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Closure under unions

- *Lemma*. Let $\langle \pi_i : i < \lambda \rangle$ be an increasing chain of wide partial types. Then $\pi = \bigcup_{i<\lambda} \pi_i$ is wide.

Minimal wide types: existence

- *Corollary* Every partial wide type can be extended to a *minimal* wide type p (a wide type who has a *unique* wide extension to any set).
- In fact, every forking extension of p is not wide.
- We would like to understand minimal wide types.

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Morley sequences

- *Definition.* Let $p \in \mathrm{S}(A)$ be a minimal wide stable type, let p^* be the unique extension of p in $\mathrm{S}(\mathcal{B})$ which is wide, and let $I = \langle a_\alpha : \alpha < \lambda \rangle$ be defined as follows: $a_\alpha \models p^* \restriction A a_{<\alpha}$. Such I is called a *Morley sequence* in p.
- *Fact.* Let I be as above, then it is an indiscernible set over A. (By stability).
- *Strong Uniqueness Lemma.* Let I be as above. Let u_i be mutually disjoint blocks of λ for $i < \omega$ and $b_i \in \sum_{\alpha \in u_i} \mathbb{R} a_\alpha$ with $\|b_i\| = 1$. Then $J = \langle b_i : i < \omega \rangle$ is an indiscernible set over A and a Morley sequence in p.

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Minimal types: analysis

- *Old Theorem* Let $r \in S_\Delta(A)$ be a minimal wide stable type, and let $I = \langle a_\alpha : \alpha < \lambda \rangle$ be a Morley sequence in r. Then I is isometric to the standard basis of ℓ_q for some $1 \leq q \leq \infty$. In other words, for every $k < \omega$ and $\lambda_0, \ldots, \lambda_{k-1} \in \mathbb{R}$, we have

$$\left\| \sum_{i<k} \lambda_i a_i \right\|^q = \sum_{i<k} |\lambda_i|^q$$

- This is, in fact, a direct consequence of the Strong Uniqueness Lemma (e.g. by a theorem of Zippin).

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$$\left\| \sum_{i<k} \lambda_i a_i \right\|^2 = \sum_{i<k} |\lambda_i|^2$$

Categoricity

- *Fact.* If K is categorical in some uncountable power, then
  - K is stable
  - Prime models exist over any set.
  - K is "uni-dimensional"
  - K is categorical in all uncountable powers.

Analysis of Categorical Classes

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Corollary. Let K be categorical. Then

- There is a "unique" (up to orthogonality) wide minimal type.
- Every $B \in K$ of uncountable density is prime over a Morley sequence in p, which is isometric to the standard basis of ℓ_2.
- Equivalently, every $B \in K$ of uncountable density is prime over a spreading model, which is isometric to ℓ_2.
- Every $B \in K$ can be seen as a reduct of an Ehrenfeucht-Mostowski model $\mathrm{EM}(\lambda, \Phi)$ where Φ is an *EM*-type of an indiscernible sequence, which is isometric to the standard basis of ℓ_2.
- Every infinite indiscernible sequence in $\mathcal{B}$ is "dominated" by a standard basis of ℓ_2.

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