Groups of finite Morley rank: current progress, current problems

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Algebricity Conjecture

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Classification of strongly minimal sets ?

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- $B < G$ is a *Full Frobenius* group, i.e. $B \cap B^g = 1$ for every g in $G \setminus B$ and $G = B^G$

Plan of the talk

- Infinite combinatorics
- Parallels with the Classification of the Finite Simple Groups
- Genericity, generosity...

Part I

INFINITE COMBINATORICS

The “bad" group problem

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Main Question: Is there a CSA-group with maximal abelian subgroups conjugate (and then $G = B^G$) and "strong" stability properties?

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What about their stability?

Independence Property

Definition (Independence Property) *A formula* $\phi(x, y)$ *has the* Independence Property *relatively to a class* $\mathcal{C}$ *of structures if for any* $n >> 1$ *there exists* $M \in \mathcal{C}$ *with tuples* $x_1, \cdots, x_i, \cdots, x_n$*, and* $y_1, \cdots, y_\sigma, \cdots, y_{2^n}$ *(*$\sigma \in 2^n$*) such that, in* M*,* $\phi(x_i, y_\sigma)$ *is true iff* $i \in \sigma$*.*

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Theorem (J - Muranov - Neman): *Let* $w(x, y)$ *be a group word. Then the probability that the formula* $w(x, y) = 1$ *has the independence property relative to the class of torsion-free (Gromov) hyperbolic groups tends rapidely to* 1 *has the length of* w *tends to the infinity.*

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Still, are there constructions for bad groups?

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Theorem (Sela 2006) F_n *has a stable theory.*

(holds more generally for torsion-free Gromov hyperbolic groups)

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New ω-stable groups ???

Part II

Parallels with CFSG

for simple groups of finite Morley rank

Morley rank

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Definability of the rank

Elimination of $\exists^\infty$ quantifer

The rank is additive. For instance, $\mathsf{rk}\,(G/H) = \mathsf{rk}\,(G) - \mathsf{rk}\,(H)$

Basic properties

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- Descending Chain Condition on definable subgroups.
- Connected component G°: smallest definable subgroup of finite index.

Connected groups

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Elementary Fact : *A connected group acting definably on a finite set fixes it pointwise.*

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Algebricity Conjecture (Cherlin - Zilber) : *An infinite simple group of finite Morley rank is algebraic over an algebraically closed field.*

Anti-Algebricity Conjecture : *There are bad groups of finite Morley rank.*

Borovik's program

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- Cyclic of prime order
- Alternating type
- Lie type
- 26 sporadics

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Borovik's Program: Use the architecture of CFSG for infinite simple groups of finite Morley rank.

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- Amalgam method.

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4^{th} generation ? ...

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- i and j involutions $\longrightarrow (ij)^i = (ij)^{-1}$
- The classification itself:
  - Char 2 type versus Char $\neq 2$ type
  - Small groups versus large groups

Sylow 2-subgroups

Theorem (Borovik - Poizat , 1990) : *Sylow 2-subgroups are conjugate and if S is one of them, then S° is nilpotent and a central product with finite intersection*

$$S^\circ = T * U$$

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- 2-*torus* : divisible abelian 2-group.
- 2-*unipotent* : definable connected 2-group of bounded exponent.

Types and characteristic

Types :

	$U \neq 1$	$U = 1$
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- Odd type $\longrightarrow G$ algebraic in characteristic $\neq 2$.
- No mixed type.
- Degenerate type ? (bad group for example)

Even type

Theorem (Altinel - Borovik - Cherlin + ...) : *A simple group of finite Morley rank of even type is algebraic over an algebraically closed field of characteristic* 2.

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Proof :

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Proof : Book □

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others

Reductivity and semisimplicity of elements of odd prime order, find root SL_2 subgroups, build an "optimistic" torus and Weyl group W, complex reflection group by ultraproduct, crystallographic Coxeter group by Shepard-Todd, recognition via Curtis-Tits-Phan

Mixed type

Theorem : *A simple group of finite Morley rank cannot be of mixed type.*

Proof

- Proceeds as if G was a direct product of a group of even type and of a group of odd type.
- Uses the classification in even type.

$\square$

Odd type

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Number of copies of $\mathbb{Z}_{2^\infty}$: *Prüfer 2-rank.*

Theorem : *A minimal counterexample to the Algebricity Conjecture of odd type has Prüfer 2-rank at most 2.*

Bad fields

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Baudisch - Hils - Martin-Pizzaro - Wagner (2006) *There exists a field $\langle K, T \rangle$ of finite Morley rank, with $T < K^{\times}$, in characteristic 0.*

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Enourmous complications! (Unipotence theory has to be developed from scratch –> Burdges Thesis)

Minimal configurations

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Call the finite center of $SL_2(K)$ an **exceptional finite set**.

“Classification" of locally$^\circ$ solvable$^\circ$ groups

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http://math.univ-lyon1.fr/∼jaligot/

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Uniqueness Theorem (J 2000 – 2007)

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Theorem (Deloro - J) A connected locally$^\circ$ solvable$^\circ$ group with involutions is either solvable, $PSL_2(K)$ with $Char(K) = 2$, or non-solvable of odd type.

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Non-solvable odd type case J 2000 - Cherlin-J 2004 - Deloro’s Thesis 2007. –> THREE PROBLEMATIC CONFIGURATIONS.

Part III

Generrix' Adventures in Groupland

Carter subgroups

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Theorem (Frécon - J 2004) : *It always exists!*

Generous Carter subgroups

Definition :

$X \subseteq_{def} G$ *is* generic *in* G *if* rk $(X) = $ rk (G).

$X \subseteq_{def} G$ *is* generous *in* G *if* X^G *is generic in* G.

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Modern geometric rank computation: X *is generous iff a generic element of* G *belongs to* finitely many *conjugates of* X *(*X *normalizing coset of a definable subgroup).*

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Generix never gives up - 2005 : *In any group of finite Morley rank,* generous *Carter subgroups are conjugate and generically disjoint.*

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Generix and the Cosets - 2009 : *The Weyl group* $N(Q)/Q$*, where* Q *is a generous Carter subgroup, acts faithfully on* Q *(almost always).*

Genericity Conjecture

Main question/conjecture : *Any group of finite Morley rank contains a generous Carter subgroup.*

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Includes algebraic groups and bad groups.
More raisonable then Cherlin's conjecture, but still very HARD!

Genericity Conjecture

Main question/conjecture : *Any group of finite Morley rank contains a generous Carter subgroup.*

Includes algebraic groups and bad groups.
More raisonable then Cherlin's conjecture, but still very HARD!

Even more food for the next decades:

Can a group have a generous Carter subgroup and a non-generous Borel subgroup?

Is there a kind of bad group with a nontrivial *Weyl group ?*