Small Polish structures

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Polish structures

Definition

A Polish structure is a pair (X, G) where G is a Polish group acting faithfully on a set X so that $G_x <_c G$ for every $x \in X$.

Examples

1. Profinite structures: X - a profinite space, G - a compact group, the action is continuous,
2. Polish G-spaces: X - a Polish space, G - a Polish group, the action is continuous, e.g.:
   - X - a compact metric space, $G = Homeo(X)$ - the group of all homeomorphisms of X with the compact-open topology,
   - X - a compact metric group, $G = Aut(X)$ - the group of all topological automorphisms of X with the c-o topology,
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Smallness

Definition

A Polish structure (X, G) is small if for every $n \in \omega$, there are only countably many orbits on X^n.

Examples of small Polish structures

1. $(S^n, Homeo(S^n))$, $n \in \omega$,
2. $(I^n, Homeo(I^n))$, $n \in \omega \cup \{\omega\}$,
3. $((S^1)^n, Homeo((S^1)^n))$, $n \in \omega \cup \{\omega\}$,
4. $(P, Homeo(P))$, P - the pseudo-arc,
5. $(H, Aut(H))$, H - a profinite abelian group of finite exponent, $Aut(H)$ - the group of all topological automorphisms of H,
6. $(H, Aut^0(H))$, H - as above, $Aut^0(H)$ - the group of all automorphisms of H preserving a distinguished inverse system indexed by ω.

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m-independence

(X, G) -a compact [profinite] structure
a -finite tuple of elements of X
A, B -finite subsets of X
$o(a/A) := \{f(a) : f \in G_A\}$

Definition

$a \overset{m}{\mathop{\smile}\limits^{|}}_A B \iff o(a/AB) \subseteq_o o(a/A)$,
$a \overset{m}{\not\mathop{\smile}\limits^{|}}_A B \iff o(a/AB) \subseteq_{nwd} o(a/A)$.

Fact (Newelski)

$\overset{m}{\mathop{\smile}\limits^{|}}$ is invariant, symmetric and transitive. If (X, G) is small, then $\overset{m}{\mathop{\smile}\limits^{|}}$ satisfies the existence of independent extensions.

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$\pi_A : G_A \to o(a/A)$ is defined by $\pi_A(g) = ga$.

Definition

$$a \mathop{\smile\!\!\!\!\!\!|}^{nm}_A B \iff \pi_A^{-1}[o(a/AB)] \subseteq_{nm} \pi_A^{-1}[o(a/A)],$$
$$a \mathop{\smile\!\!\!\!\!\!|}^{m}_A B \iff \pi_A^{-1}[o(a/AB)] \subseteq_{m} \pi_A^{-1}[o(a/A)].$$

Theorem

$\mathop{\smile\!\!\!\!\!\!|}^{nm}$ is invariant, symmetric and transitive. If (X, G) is small, then $\mathop{\smile\!\!\!\!\!\!|}^{nm}$ satisfies the existence of independent extensions.

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$\downarrow^{nm}$ is invariant, symmetric and transitive. If (X, G) is small, then $\downarrow^{nm}$ satisfies the existence of independent extensions.

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In compact [profinite] structures, $\downarrow^{nm} = {}^{m}\!\downarrow$.

$\mathcal{NM}$-rank and *nm*-stability

(X, G) - a small Polish structure
$\mathcal{NM}$: orbits over finite sets $\rightarrow Ord \cup \{\infty\}$

$\mathcal{NM}(a/A) \geq \alpha + 1$ iff there is a finite set $B \supseteq A$ such that $a \not\downarrow^{nm}_A B$ and $\mathcal{NM}(a/B) \geq \alpha$.

Definition

$\mathcal{NM}(X) = \sup\{\mathcal{NM}(x) : x \in X\}$.

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(X, G) is *nm*-stable if for every $x \in X$, $\mathcal{NM}(x) < \infty$.

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2. In $(I^\omega, Homeo(I^\omega))$, $\mathcal{NM}(I^\omega) = 1$.
3. $(P, Homeo(P))$ where P is the pseudo-arc is not *nm*-stable.
4. In $((\mathbb{Z}_{p^n})^\omega, Aut((\mathbb{Z}_{p^n})^\omega))$, $\mathcal{NM}((\mathbb{Z}_{p^n})^\omega) = n$.

Remark

Everything that was said about $\mathop{\smile\!\!\!\!|}^{nm}$ works in a suitably defined imaginary extension X^{eq} of X (e.g. if G acts continuously on a space X and E is an invariant, closed equivalence relation on X^n, then $X^n/E \subseteq X^{eq}$).

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Compact G-groups

Definition

A Polish [compact] G-group is a Polish structure (H, G) where G acts continuously and by automorphisms on a Polish [compact] group H.

Main examples of compact G-groups

$(H, Aut(H))$ where H is a compact metric group.

Definition

A profinite group regarded as profinite structure is a profinite structure (H, G) where H is a profinite group and G acts on it by automorphisms. In particular, G is also a profinite group.

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Basic remarks

Remark

Each profinite group (H, G) is a compact G-group.

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Remark

If H is a profinite group, then $Aut(H)$ is not necessarily compact, so $(H, Aut(H))$ needn't be a profinite structure.

Basic remarks

Remark

1. If (H, G) is a small Polish G-group, and $S \subseteq H$ is finite, then $\overline{\langle S \rangle}$ is countable, i.e. $\langle S \rangle$ does not have limit points.
2. If (H, G) is a small compact G-group, then H is locally finite.

Example

Consider the additive structure on $\mathbb{Q}$. Take the discrete topology on $\mathbb{Q}$ and the product topology on $\mathbb{Q}^\omega$. For a suitably chosen Polish group G, $(\mathbb{Q}^\omega, G)$ is a small Polish G-group of $\mathcal{NM}$-rank 1, which is torsion-free and 0-dimensional.

Problem

Find non 0-dimensional small Polish G-groups.

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Conjectures on small profinite groups

Proposition

If (H, G) is a small compact G-group, then H is a profinite group. But G is not necessarily compact, so (H, G) needn't be a small profinite group.

(H, G) - a small profinite group

Newelski's Conjecture

H is abelian-by-finite.

Intermediate Conjectures

(A) H is solvable-by-finite.

(B) If H is solvable-by-finite, then it is nilpotent-by-finite.

(C) If H is nilpotent-by-finite, then it is abelian-by-finite.

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Results on small profinite groups

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Each small, nm-stable profinite group is abelian-by-finite.

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The conjectures for small compact G-groups

From now on, we consider generalizations of Conjectures (A), (B), (C) to the wider context of small compact G-groups. It turns out that in general they are all false.

Counter-example to (A)

H - any finite non-solvable group
S_∞ - the group of all permutations of $\mathbb{N}$
S_∞ acts on H^ω by $\sigma\langle h_0, h_1, \dots\rangle = \langle h_{\sigma(0)}, h_{\sigma(1)}, \dots\rangle$.

Then (H^ω, S_∞) is a small compact S_∞-group, and H^ω is not solvable-by-finite.

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Results on small, nm-stable compact G-groups

(H, G) - a small compact G-group

Theorem

If (H, G) is nm-stable, then H is nilpotent-by-finite.

Theorem (K., Wagner)

If $\mathcal{NM}(H) \leq \omega$, then H is abelian-by-finite.

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Infinite ordinal $\mathcal{NM}$-ranks are possible

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- Choose $\omega = I_0 \supseteq I_1 \supseteq \dots$ so that $I_i \setminus I_{i+1}$ are all infinite and $\bigcap I_i = \emptyset$,
- $H_i := \{\eta \in \mathbb{Z}_p^\omega : \eta(j) = 0 \text{ for all } j \in I_i\}$, $i \in \omega$,
- $G := \{g \in Aut(\mathbb{Z}_p^\omega) : g[H_i] = H_i \text{ for every } i \in \omega\}$.

Then:

1. $(\mathbb{Z}_p^\omega, G)$ is a small compact G-group,
2. $\mathcal{NM}(\mathbb{Z}_p^\omega) = \omega$,
3. $H_0 <_{nwd} H_1 <_{nwd} \dots$ is an infinite increasing sequence of $\emptyset$-closed subgroups.

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Open questions on small Polish G-groups

Question

Is every small, *nm*-stable Polish G-group abelian-by-countable?

Even the following question is open.

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Is every small Polish G-group of $\mathcal{NM}$-rank 1 abelian-by-countable?

Future investigations of Polish structures

- Prove counterparts of some deep model theoretic results.
- Find further examples, especially of small Polish G-groups; understand which compact metric spaces with the full group of homeomorphisms are small.
- Investigate the structure of groups and rings.
- Try to apply the introduced model theoretic tools to purely descriptive set theoretic problems.
- Try to find a notion of interpretability in first order structures.