Fine Structure of Dependence in Superstable Theories of Finite Rank

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The Question

Models of one-based theories satisfy:

$$\forall a, b \quad a \not\mathop{\smile\hspace{-0.9em}\mid} b \Longrightarrow \operatorname{acl}(a) \cap \operatorname{acl}(b) \neq \operatorname{acl}(\emptyset)$$

Non-locally modular strongly minimal sets do not satisfy this property. But the question remains whether in a superstable theory of finite rank this is the only way in which the property can fail. In other words, we can ask whether the property holds of sets arising in a more complex way via a level construction with semiminimal sets, the "building blocks" of a superstable theory of finite rank.

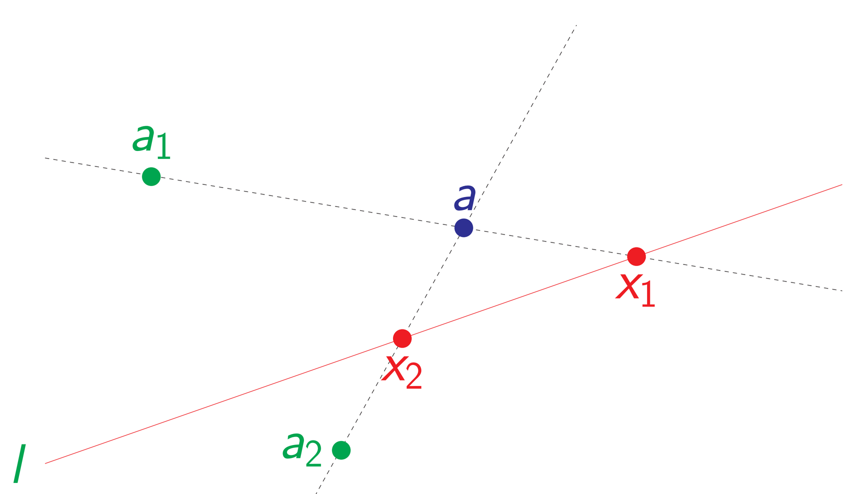
Semiminimal-Construction of Models

Definition. *Let q be a minimal type and p a nonalgebraic strong type. p is q-semiminimal if for some set A over which both p and q are based, there are $a \models p|A$ and $c_0, \ldots, c_n \models q|A$ such that*

$$a \in \operatorname{acl}(c_0, \ldots, c_n, A)$$

p is semiminimal if it is q-semiminimal for some q. A set of realizations of a semiminimal type is called a semiminimal set.

Example. $\mathbb{P}^2$ is semiminimal. Let a be a generic point, $p = \operatorname{stp}(a)$; l a line; $a_1 \neq a_2$ points independent from a that do not lie on l, and X the set of points on the line l. Pick x_1, x_2 in X as in the picture:



Letting $A = \{l, a_1, a_2\}$ we have $a \in \operatorname{acl}(x_1, x_2, A)$. Thus, $\mathbb{P}^2$ is semiminimal.

Let C denote the sequence $\{c_i : i < \alpha\}$, and $C_{<j} := \{c_i : i < j\}$. Given a set A, C is a *semiminimal-construction over A* if for each $i < \alpha$, $\operatorname{stp}(c_i/C_{<i} \cup A)$ is semiminimal or algebraic.

Lemma. *In a superstable theory, for any set B there is an enumeration of $\operatorname{acl}(B)$ that is a semiminimal-construction. If B is of finite rank, there is an $n \leq U(B)$, $b = (b_1, \ldots, b_n)$, with $\operatorname{stp}(b_i/b_{<i})$ semiminimal for $1 \leq i \leq n$ and $\operatorname{acl}(B) = \operatorname{acl}(b)$.*

Partitioning a Semiminimal-Construction into Levels

Due to the Lemma above, a partitioning of semiminimal-constructions can be given in terms of levels, defined as follows:

Definition. (i) *For sets A and B, the first level of A over B, written $l_1(A/B)$, is $\{a \in \operatorname{acl}(A) : \operatorname{stp}(a/B)$ is semiminimal or algebraic$\}$.*

(ii) *For a set C and $k \geq 0$, the k^{th} level of C, written $l_k(C)$, is defined recursively:*

(a) $l_0(C) = \emptyset$.

(b) $l_{n+1}(C) = l_1(C/l_n(C)) \cup l_n(C)$.

We say *C has m levels* if m is the least k for which $C \subseteq \operatorname{acl}(l_k(C))$.

The Conjecture

Any element a of a model of a superstable theory of finite rank has a semiminimal-construction. Given an element a we define $\underline{a}$ to be some $d \in \operatorname{acl}(a)$ such that a is interalgebraic over d with a sequence of elements from semiminimal sets. The property in question can be characterized as follows:

Level Dependence Property (LDP). If a is an element satisfying $\underline{a} \neq \emptyset$ and b is some element such that a depends on b over $\underline{a}$, then $\operatorname{acl}(\underline{a}) \cap \operatorname{acl}(b) \neq \operatorname{acl}(\emptyset)$.

We conjecture that LDP holds in all superstable theories of finite rank. This formalizes the intuition that the only way dependence does not imply algebraic dependence is if a has one level.

Equivalent Versions of LDP

We show that LDP is equivalent to the Canonical Base Property (CBP) introduced by Moosa and Pillay (2008). That the Canonical Base Property (CBP) is equivalent to a property we refer to as Chatzidakis' Property is already known [Chatzidakis, 2008].

As in [Moosa-Pillay, 2008], $\mathbb{P}$ below denotes the set of all nonmodular minimal types.

Proposition. *The following are equivalent:*

- *(LDP) If a is an element satisfying $\underline{a} \neq \emptyset$ and b is some element such that a depends on b over $\underline{a}$, then $\operatorname{acl}(\underline{a}) \cap \operatorname{acl}(b) \neq \operatorname{acl}(\emptyset)$.*
- *(CBP) If $b = \operatorname{Cb}(a/b)$, then $\operatorname{stp}(b/a)$ is almost $\mathbb{P}$-internal.*
- *(Chatzidakis' Property) If $b = \operatorname{Cb}(a/b)$, then $\operatorname{stp}(b/\operatorname{acl}(a) \cap \operatorname{acl}(b))$ is almost $\mathbb{P}$-internal.*

Motivation

LDP and CBP, using different methods, attempt to generalize the following kind of phenomenon from compact complex spaces to superstable theories of finite rank:

Theorem (Pillay 2002). *Every minimal type in an ω-saturated model of the compact complex varieties is either locally modular or nonorthogonal to the generic type of the projective line.*

Similar dichotomies have been obtained in differential and difference fields.

Some More Definitions

Before stating our results, we need:

Definition. (i) *An element a is unidimensional if any two semiminimal types appearing in the semiminimal-construction of a are nonorthogonal.*

(ii) *A maximal collection $(b_0, \ldots, b_n) \in \operatorname{acl}(b)$ of unidimensional elements are called the unidimensional coordinates of b.*

(iii) *An element a is called minimal with respect to b if a depends on b over $\underline{a}$ and has minimal U-rank among all elements in $\operatorname{acl}(a)$ that depend on b over $\underline{a}$.*

Results

Main Theorem. *Let a and b be such that a depends on b over $\underline{a}$ and $\operatorname{acl}(\underline{a}) \cap \operatorname{acl}(b) = \operatorname{acl}(\emptyset)$. Then, there is a unidimensional $c \in \operatorname{acl}(a)$ that depends on b over $\underline{c}$ and $\operatorname{acl}(\underline{c}) \cap \operatorname{acl}(b) = \operatorname{acl}(\emptyset)$.*

In particular, the above shows that LDP holds for any nonunidimensional a that is minimal with respect to b.

LDP also holds under certain assumptions on ranks:

Second Theorem. *Let a be an element satisfying $\underline{a} \neq \emptyset$ and b be an element such that a depends on b over $\underline{a}$. If $\operatorname{U}(a/\underline{a}) = 1$ or $\operatorname{U}(a/\underline{a}) = 2 = \operatorname{U}(b/\underline{b})$, then $\operatorname{acl}(\underline{a}) \cap \operatorname{acl}(b) \neq \operatorname{acl}(\emptyset)$.*

As a consequence of the Main Theorem we obtain:

Corollary. *Let a and b be elements such that a is minimal with respect to b, $\underline{a} \neq \emptyset \neq \underline{b}$, with $(a_0, \ldots, a_n)$ the unidimensional coordinates of a, and $c = \operatorname{acl}(\underline{a}) \cap \operatorname{acl}(\underline{b})$. Then*

$$a \mathop{\smile\hspace{-0.9em}\mid}_{\underline{a}a_0 \ldots a_n c} b$$

Proofs of the above results make heavy use of canonical bases as well as the fine analysis of dependence between levels resulting from the level partitioning of semiminimal-constructions.

Remaining Case

The remaining case of LDP for a unidimensional a seems to require new ideas. We think that the level partitioning of semiminimal-constructions will continue to aid in our analysis. Perhaps it would be useful to find an alternate characterization of LDP that is in a way more "internal" to the framework.

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