ESF SHORT VISIT GRANT REPORT

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Purpose of the visit. Last April I had the opportunity to start a research project on the Random Conductance Model (RCM) together with Prof. Marek Biskup and Tilman Wolff. My first stay in Los Angeles, which was also funded by an RGLIS grant, gave as a result the article *A central limit theorem for the effective conductance: I. Linear boundary data and small ellipticity contrasts*, available on the arXiv (reference number 1210.2371). The aim of my visit in October was to generalise the results previously obtained in the research project.

Description of the work. The RCM consists in assigning to every edge of the square d-dimensional grid a random weight sampled from a, say, i.i.d. distribution. The *effective conductance* on a finite volume box Λ_L with side L is the minimum of the *Dirichlet energy* of the system with prescribed boundary condition. For conductances $\{\omega_{xy}\}_{x\sim y}$ and linear boundary conditions $t \in \mathbb{R}^d$, this is

$$Q_L^t(\omega) = \inf\Big\{ \sum_{x\sim y,\, \{x,y\}\cap\Lambda_L\neq\emptyset} \omega_{x,y}\big(f(y)-f(x)\big)^2 \ \text{ s.t. } f(z) = t\cdot z \quad \forall z \in \partial\Lambda_L \Big\}.$$

It is well known that Q_L^t divided by the size of the domain L^d converges, as the box grows invading the whole lattice, to a non-random constant. This limit has been explicitly characterized in the eighties by, among the others, Papanicolau and Varadhan, Kozlov, Künnemann.

The variance of such a random variable is harder to handle. Its correct order has been proved to be L^d only last year by Gloria and Otto in the article *An optimal variance estimate in stochastic homogenization of discrete elliptic equations.*

In our paper, we managed to describe the gaussian nature of the oscillations of the effective conductance around its mean, proving a Central Limit Theorem. Unfortunately, we had to assume conductances with small ellipticity contrast, i.e., they must be close to a constant for our technique to work (in particular, we need it when applying the Meyers' estimates in order to control the moments of the harmonic corrector). This setting resembles very well the nature of physical materials with irregular microscopic structure, but is unsatisfactory from a mathematical point of view.

The main task of this two weeks in Los Angeles has been to try removing this assumption.

In the meantime, similar results have appeared in the literature (Nolen, Rossignol). Nevertheless, the authors address in their works the case of periodic boundary conditions, less appealing from a physical point of view.

Description of the main results obtained. The main issue when dropping the small ellipticity contrast condition is how to control the moments of the (gradients of the) finite and infinite volume harmonic corrector in the case of elliptic conductances. One of the possible paths to pursue is the strategy presented in the article by Gloria and Otto mentioned above. They are in fact able to prove the boundedness of the q-th moment of the infinite harmonic corrector (and therefore of its gradient) for any $q \in \mathbb{R}^+$ and in any dimension $d \geq 3$. Retracing their proof and adapting it to the finite volume case, one can prove the same statement for the finite volume harmonic corrector. This is exactly what is needed for proving the Central Limit Theorem in larger generality. However, this strategy has its drawbacks:

- It does not allow one to prove the CLT with elliptic conductances in dimension 2;
- The proof of Gloria and Otto is very long and technical: reproducing it in the finite volume case would not bring any original contribution to the field.

We followed therefore alternative techniques to attack the problem. The first one was to try controlling the growth of the moments of the finite volume corrector when the value of the conductances is slightly changed. This growth can be explicitly described when only one of the conductances is raised. Interestingly, one has an explicit control of the gradient of the harmonic corrector in each of its components. Things become way more complicated when more conductances are modified at the same time and we lose some concavity property that shows up when only one conductance value is changed.

Another possible line of investigation is to use Gloria and Otto's bound of the infinite corrector's moments in order to deduce a similar bound for the finite volume object. In this case we are able to bring the problem back to (apparently easier) questions of non-exiting probabilities from the given box for random walks among random conductances, which we plan to study in greater detail in the immediate future.

To summarize, we were not able to overcome the big obstacle of general elliptic conductances on the two dimensional lattice. Nevertheless, our research brought many intermediate results (such as explicit formulas for the derivative and second derivative of the double gradient of the random Green's function, useful formulas for the reduction of complex networks to simpler ones and others) and opened many paths to follow in the future.

Future collaboration with host institution. The collaboration with Prof. Biskup has turned out once more to be extremely positive and motivating. We definitely plan to continue research on this project and meet again as soon as there will be the chance (hopefully in Europe whithin the end of the present year).

Other comments. Once more I would like to thank ESF for the support, especially for answering promptly to a late request. The results obtained in this period of research will enrich my Ph.D. thesis which I plan to defend next year. This experience strenghtens the fruitful bridge between the Probability Theory communities of Berlin and Los Angeles and therefore, in small measure, between Europe and the rest of the world.

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