Scientific report - Weak solutions of equations associated with (non-local) Dirichlet forms

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Time and place of visit.

The visit in Japan from February 11th until March 8th 2014 included stays at the institutes Tohoku University Sendai (February 11th - 25th and March 3rd - 8th) and Kansai University Osaka (Febrary 26th - March 2nd). While the whole trip was co-financed by various sources the ESF supported the first period of the stay at Sendai (February 11th - 25th) next to parts of the overall traveling costs.

Purpose of the visit.

The main purpose of the visit to Sendai was to collaborate with Jun Masamune on an ongoing project (of Daniel Lenz, Matthias Keller, Jun Masamune and myself). The goal of this project is to fully understand the relation of global properties of Dirichlet forms and of properties of weak solutions to equations associated with them.

Description of the work carried out during the visit and main results obtained.

The work carried out during the visit can be divided into two parts.

In the first part we generalized the main result of [1] from the weighted graph case to arbitrary regular Dirichlet forms. Namely, we showed that uniqueness of bounded weak solutions to the heat equation is equivalent to a generalized conservation property ('stochastic completeness at infinity') of the associated L^{∞}-semigroup. We obtained further abstract characterizations of stochastic completeness at infinity in terms of bounded weak solutions to the Laplace equation and in terms of the existence of suitable cut-off functions. Here, our particular focus lay on the situation when the killing measure of the Dirichlet form does not vanish and/or the Dirichlet form is non-local.

As a second part we applied the developed theory to a Schrödinger operator on a Riemannian manifold with positive potential. In this situation the generalized conservation property takes the form

$$1 = e^{t(\Delta - V)}1 + \int_0^t e^{s(\Delta - V)} V ds, \tag{1}$$

where Δ is the Dirichlet Laplacian and $V \geq 0$ the potential. In the case of model manifolds with radially symmetric V we could then fully characterize the validity of (1) in terms of volume growth and the growth of V. For general Riemannian manifolds we obtained sufficient conditions for the validity of (1) in terms of volume growth with respect to a metric which may be strictly larger than the Riemannian path metric.

Future Collaborations and Project Publications.

The results mentioned above deal with weak solutions in L^∞. However, many of the techniques we used work without the boundedness assumption. One particular aspect we want to investigate in the future are criteria for non-local Dirichlet forms to have unique weak L^2 solutions. This will yield essential self-adjointness results for the corresponding operators.

The work on stochastic completeness at infinity will result in a publication and support of the ESF will be gratefully acknowledged.

[1] M. Keller, D. Lenz, Dirichlet forms and stochastic completeness of graphs and subgraphs, J. Reine Angew. Math., Volume 2012, Issue **666**, 189–223.