ESF- Science Meeting - Scientific report

Fall School Phase Transition in Random Discrete Structures

September 02 - 20, 2013
Graz University of Technology, Austria

1 Summary

The fall school gave advanced master students, PhD students, and early stage postdocs, who are interested in random discrete structures and related fields, the opportunity to

- learn the state of the art results in the study of the phase transition in various random discrete structures;
- to understand modern proof techniques that have successfully been applied to the study of phase transition and its critical behaviour;
- to meet fellow colleagues in their early research career, which possibly results in their future collaboration.

The focus of the school was on the recent results of Amin Coja-Oghlan and Konstantinos Panagiotou. In particular they developed a new Survey Propagation and a new asymmetric second moment method for random Constraint Satisfaction Problems (CSPs). They have verified the statistical mechanics conjecture for a special case of random satisfiability problems. They also have several other important results in the phase transition in random graphs and random graph processes. During the course the following topics were discussed:

- Connection between statistical physics and random CSP problems
  by Amin Coja-Oghlan (Goethe University of Frankfurt)
- 2-coloring of random uniform hypergraphs
  by Amin Coja-Oghlan (Goethe University of Frankfurt)
- The Giant Component in Random Graph Processes
  by Konstantinos Panagiotou (Ludwig Maximilian University of Munich)
- Orientability Thresholds for Random Hypergraphs
  by Konstantinos Panagiotou (Ludwig Maximilian University of Munich)
- The k-NAESAT Threshold
  by Konstantinos Panagiotou (Ludwig Maximilian University of Munich).

The course consisted of lectures and solving problem sets assigned by the lecturer in small groups. In addition the solutions of the problem sets were discussed by the whole group every second day.

The final day of the fall school consisted of a mini-workshop on phase transitions on random graphs. The plenary speaker was Oliver Riordan (University of Oxford) and the invited speakers were Oliver Cooley (Graz University of Technology) and Charilaos Efthymiou (University of Frankfurt). In addition short talks were presented by some of the participants.

2 Scientific Content

The aim of the course was to discuss recent results in random CSP problems and the phase transition in random graphs and random graph processes. The following topics were covered during the course:

- Connection between statistical physics and random CSP problems
- 2-coloring of random uniform hypergraphs
- The Giant Component in Random Graph Processes
- Orientability Thresholds for Random Hypergraphs
- The k-NAESAT Threshold.

The first section of the course discussed *The connection between statistical physics and random CSP problems*. The course was presented by Amin Coja-Oghlan (Goethe University of Frankfurt) in two 2 hour lectures.

Random discrete structures play an important role in statistical mechanics as mathematical models (mean-field models) of disordered systems. Physicists have developed the cavity method (message passing procedures), which has successfully been applied to the study of phase transitions (e.g. their location and nature) and has led to a sophisticated but non-rigorous formalism for the notorious random satisfiability problems and random constraint satisfaction problems. This formalism yields conjectures on the threshold values of many random CSPs.

In particular the conjecture for the 2-colorability of random hypergraphs were discussed. A rigorous proof for the conjecture was given in the following section *The 2-coloring of random uniform hypergraphs* was presented by Amin Coja-Oghlan (Goethe University of Frankfurt) in three 2 hour lectures.

Select a hypergraph uniformly at random from the set of K-uniform hypergraphs on N vertices and M edges for some fixed $K \geq 3$. Let $N \to \infty$ and $M/N = \alpha$ for some fixed α. An event $\mathcal{A}$ holds with high probability (w.h.p.) if $\mathbb{P}(\mathcal{A}) \to 1$ as $N \to \infty$. A hypergraph is 2-colorable if its vertices can be colored by two colors such that no edge is monochromatic. The conjecture is that the threshold for 2-colorability is

$$\alpha_0 = 2^{K-1} \ln 2 - \left(\frac{\ln 2}{2} + \frac{1}{4} \right)$$

i.e. when $\alpha < \alpha_0$ then the randomly chosen hypergraph is w.h.p. 2-colorable and when $\alpha > \alpha_0$ then the randomly chosen hypergraph is w.h.p. not 2-colorable

The third section was on *The Giant Component in Random Graph Processes* presented by Konstantinos Panagiotou (Ludwig Maximilian University of Munich) in three 2 hour lectures.

The random graph processes considered here were Achlioptas processes. In particular we considered the bounded product rule in Achlioptas processes. Achlioptas suggested the following extension of the Erdős-Rényi random graph process in order to delay the emergence of the giant component. The Erdős-Rényi random graph process starts out from the empty graph on n vertices and in every step selects and edge uniformly at random from the set of edges not present in the graph and inserts it into the graph. The process stops after m steps. In the Achlioptas process instead of selecting an edge at random and inserting it into the graph multiple edges are selected, each of them uniformly at random from the set of edges which have not yet been inserted into the graph, and then a choice is made according to a fixed rule which of the edges is inserted into the graph. The rule can only depend on the previously inserted edges. The product rule has received a lot of attention as it is believed to delay the emergence of the giant component the most. Let $c(v)$ denote the size of the component in which the vertex v is found. In the product rule one selects 2 pairs of random vertices $\{v_1, v_2\}, \{v_3, v_4\}$ and inserts the edge $\{v_1, v_2\}$ if $c(v_1)c(v_2) < c(v_3)c(v_4)$ otherwise one inserts the edge $\{v_3, v_4\}$. The bounded version of this process is reached by treating all of the vertices in components larger than a constant C in the same way. The proof for determining the threshold for the giant component in Achlioptas processes with bounded product rule was shown.

The fourth section considered *Orientability Thresholds for Random Hypergraphs* presented by Konstantinos Panagiotou (Ludwig Maximilian University of Munich) in two 2 hour lectures.

A hypergraph is orientable if there exists an injective function $h : E \to V$ such that for every edge $e \in E$ we have that $h(e) \in e$. The course outlined the proof for the threshold for orientability for k-uniform random hypergraphs.

In the final section *The k-NAESAT Threshold* was presented by Konstantinos Panagiotou (Ludwig Maximilian University of Munich) in two 2 hour lectures.

Let $V = \{v_1, ..., v_n\}$ be a set of boolean variables then F is a k-SAT formula if

$$F = \bigwedge_{i=1}^{m} \bigvee_{j=1}^{k} \ell_{i,j}$$

where $\ell_{i,j} = v$ or $\ell_{i,j} = \overline{v}$ for some $v \in V$. Let $\sigma : V \to \{0, 1\}$ be an assignment of the variables. We say that σ satisfies F if

$$\bigwedge_{i=1}^{m} \bigvee_{j=1}^{k} \sigma(\ell_{i,j})$$

is true. The assignment σ NAE satisfies F if both σ and $\overline{\sigma}$ satisfy F. Let $F_{n,m}$ be a k-SAT formula chosen uniformly at random from the set of k-SAT formulas with n variables and m clauses. During the course the proof for the threshold for random k-NAE satisfiability was presented.

The mini-workshop consisted of the following talks:

- *Asymptotic normality of the giant component in random hypergraphs*
  Oliver Riordan (University of Oxford)
  In this talk results on the size and structural properties of the largest component in the binomial model of the random k-uniform hypergraph were presented.

- *The Giant Component in Random Hypergraphs*
  Oliver Cooley (Graz University of Technology)
  For $1 \le j \le k-1$ two j-sets J_1 and J_2 in a k-uniform hypergraph are j-connected if there exist a sequence of edges $E_1, ..., E_\ell$ such that $J_1 \subset E_1$ and $J_2 \subset E_\ell$ and $|E_i \cap E_{i+1}| \ge j$. A giant j-connected component consists of $\Theta(n^j)$ edges. In this talk the phase transition thresholds for the appearance of the giant component in the binomial model of a random k-uniform hypergraph were determined for every $1 \le j \le k-1$.

- *MCMC sampling colourings and independent sets of G(n,d/n) near the uniqueness threshold*
  Charilaos Efthymiou (Goethe University Frankfurt)
  Sampling from Gibbs distribution is a central problem in computer science as well as in statistical physics. The focus of the talk was on the k-colouring model and the hard-core model with fugacity λ when the underlying graph is an instance of the Erdős-Rényi random graph with average degree d where d is fixed. Use the Markov Chain Monte Carlo method for sampling from the aforementioned distributions. In particular, consider Glauber (block) dynamics. In the talk a dramatic improvement was shown on the bounds for rapid mixing in terms of the number of colours and the fugacity for the corresponding models.

- *Localization in Random Geometric Graphs with Too Many Edges*
  Matan Harel (Courant Institute)
  Consider a random geometric graph, given by connecting two vertices of a Poisson Point Process of intensity n on the unit torus whenever their distance is smaller than the parameter $r(n)$ and condition on the rare event that the number of edges observed, is greater than $[1 + \delta(n)]\mathbb{E}(|E|)$. In the talk it was shown that depending on $\delta(n)$ the excess edges will result either from the appearance of a "giant

clique" or from the entropy-like effects of vanishing Radon-Nikodyn derivatives. Also the threshold function between the two behaviors was established.

- *Size of the largest component in a multi-type generalization of Erdős-Rényi random graphs*
  Christoph Koch (Graz University of Technology)
  Consider the random graph where there are two types of vertices. The vertex pair $\{u, v\}$ is connected with probability $p_{1,1}, p_{1,2}$ or $p_{2,2}$ depending on the type of u and v. In the talk the size of the largest component was analyzed in the weakly supercritical region.

- *Random Lifts of Graphs*
  Marcin Witkowski (Adam Mickiewicz University)
  We say that H is a lift of G if there exists a map $\pi : V(H) \to V(G)$ such that for every $v \in V(H)$ the restriction of π to the neighborhood of v is a bijection onto the neighborhood of $\pi(v) \in V(G)$. Moreover, if for every $v \in V(G)$ we have that $|\pi^{-1}(v)| = n$ then H is an n-lift of G. The random n-lift of a graph G is obtained by selecting uniformly at random a graph from the set of n-lifts of G. In the talk several properties of random n-lifts were discussed e.g. expander, connectivity and Hamiltonicity.

3 Impact of the event

The lecture courses were given by first rate experts in the given field. During the courses the state of the art results in the study of the phase transition in various random discrete structures and the modern proof techniques developed in order to study these phase transition have been presented.

Beside learning new ideas and techniques, the collective activities, such as solving problems in small groups and the exercise sessions, have allowed a positive interaction between the participants and gave them the opportunity to strengthen and consolidate relations with researchers from different groups and universities.

During the mini-workshop taking place at the end of the school further results on the phase transitions in random graphs and random hypergraphs were presented enabling the students to extend their knowledge on the recent developments and open questions in the field.

4 Programme of the meeting

Week 1

Time	Monday (2.9)	Tuesday (3.9)	Wednesday (4.9)	Thursday (5.9)	Friday (6.9)
9:00-9:20	Registration				
9:20-10:00	Opening				
10:00-12:00	Lecture	Lecture	Lecture	Lecture	Lecture
12:00-14:00	Lunch	Lunch	Lunch	Lunch	Lunch
14:00-16:00	Problem solving	Problem solving	Problem solving	Problem solving	Problem solving
16:00-18:00	Reception	Group discussion	Exercise session	Group discussion	Exercise session

Week 2

Time	Monday (9.9)	Tuesday (10.9)	Wednesday (11.9)	Thursday (12.9)	Friday (13.9)
10:00-12:00	Lecture	Lecture	Lecture	Lecture	Lecture
12:00-14:00	Lunch	Lunch	Lunch	Lunch	Lunch
14:00-16:00	Problem solving	Problem solving	Problem solving	Problem solving	Problem solving
16:00-18:00	Group discussion	Group discussion	Exercise session	Group discussion	Exercise session

Week 3

Time	Monday (16.9)	Tuesday (17.9)	Wednesday (18.9)	Thursday (19.9)	Friday (20.9)
10:00-12:00	LL Seminar	LL Seminar	Lecture	Lecture	Miniworkshop
12:00-14:00			Lunch	Lunch	
14:00-16:00			Problem solving	Problem solving	
16:00-18:00			Group discussion	Exercise session	

The annual Leoben-Ljubjana graph theory seminar took place in Graz this year. The seminar took place on the 16th and 17th of September. No programs were scheduled by the fall school on these days in order to enable the students to attend the talks at the seminar.

5 Full list of speakers and participants

Main Speakers:

Amin-Coja Oghlan (Goethe University, Frankfurt am Main)

Konstantinos Panagiotou (LMU Münich)

Plenary Speaker (Mini-Workshop):

Oliver Riordan (University of Oxford)

Invited Speakers (Mini-Workshop):

Charilaos Efthymiou (Goethe University Frankfurt am Main)

Oliver Cooley (Graz University of Technology)

Organizers:

Mihyun Kang

Tamás Makai

Philipp Sprüssel

Internal Participants(total 8):

Tetiana Boiko
Oliver Cooley
Johannes Cuno
Florian Greinecker
Christoph Koch
Dijana Kreso

Milton Minervino
Angélica Pachón

External Participants (total 19):

Eleni Bakali (University of Athens)
Marie-Louise Bruner (Technische Universität Wien)
Jennifer Dempsey (University of Arizona)
Hafsteinn Einarsson (ETH Zürich)
Alexey Gronskiy (ETH Zürich)
Klaas Hagemann (Leibniz University Hannover)
Matan Harel (Courant Institute)
Jan Hladký (University of Warwick)
Caha Libor (Comenius University)
Emanuele Natale (University of Rome)
Rajko Nenadov (ETH Zürich)
Diana Piguet (University of Birmingham)
Ali Pourmiri (Max Planck Institute)
Marko Savić (University of Novi Sad)
Fiona Skerman (University of Oxford)
Benedikt Stufler (LMU Münich)
Panagiotis Theodoropoulos (University of Athens)
Dominik Vu (University of Memphis)
Marcin Witkowsi (Adam Mickiewicz University)