ESF EXCHANGE VISIT GRANT REPORT

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Purpose of the visit. The aim of my five-weeks visit to UCLA was to investigate certain features of the *Random Conductance Model* (RCM) and the closely related process *Random Walk among Random Conductances* (RWRC). The collaboration with Prof. Marek Biskup brought to relevant progress in our starting problem, namely, the derivation of large deviations for the local times of RWRC forced in a small growing box, and also opened extremely interesting new subjects of research. In particular, we focused on the fluctuations of the Dirichlet form of the RCM around its mean when we take a fine discretization of the correspondent continuous homogenized model. Both these works are intended to be included in my Ph.D. Thesis.

I also attended the MSRI conference *Random Walks and Random Media* in Berkely in the last week of my stay. I had then the opportunity to follow the most recent results obtained from the most influent world experts in my field.

Description of the work. Recall that the RWRC is a particular kind of *Random Walk in Random Environment* (RWRE). The environment consists in non-negative random weights $(\omega_{x,y})_{x\sim y\in\mathbb{Z}^d}$, called *conductances*, assigned to each nearest-neighbour bond $\{x,y\}$ of the $\mathbb{Z}^d$ lattice (from here on we will consider them to be i.i.d. with some given distribution). The random walk $(X_t)_{t\geq 0}$ starts in zero and chooses its next position among its neighbours with probability proportional to the conductance of connecting bond. In its continuous version, the RWRC is driven by the *randomly-perturbed Laplace operator* Δ^ω defined by

$$\Delta^\omega f(x) := \sum_{y\in\mathbb{Z}^d\colon y\sim x} \omega_{xy}(f(y)-f(x)), \qquad f\colon \mathbb{Z}^d\to\mathbb{R},\, x\in\mathbb{Z}^d.$$

In the time spent in UCLA we faced two different (but strictly connected) aspects of this model.

1. The first feature is the behaviour of the local times of the process, defined as $\ell_t(x)=\int_0^t \mathbb{1}_{\{X_s=x\}}\mathrm{d}s$, for $x\in\mathbb{Z}^d$. We ask ourselves what the *annealed* (i.e., averaged w.r.t. the conductances) probability for ℓ_t to assume a given deterministic shape is, if we condition the random walk to stay in a box of a certain size α_t. The choice $\alpha_t = t^{1/(d+2)}$ gives an

interesting reciprocal interaction between the environment law and the typical behaviour of the underlying simple random walk.

2. The second project deals with the *Dirichlet Energy* of the RCM. Given a box Λ_L centered in the origin with side length $2L$, define the Dirichlet Energy in Λ_L with linear boundary conditions t for a configuration of conductances $\omega = (\omega_{x,y})_{x\sim y\in\mathbb{Z}^d}$ as

$$Q_L^t(\omega) = \inf\Big\{ \sum_{x\sim y,\,\{x,y\}\cap\Lambda_L\neq\emptyset} \omega_{x,y}\big(f(y)-f(x)\big)^2 \text{ s.t. } f(z) = t\cdot z \quad \forall z\in\partial\Lambda_L\Big\},$$

where t is a vector in $\mathbb{R}^d$ and $\partial\Lambda_L$ the boundary points of the box. From [GO11] and [W97] it is known that Q_L^t grows as L^d when the conductances are *strongly elliptic*, that is, bounded away from zero and infinity. The natural open question that was still open was the behaviour of the fluctuations of Q_L^t around its mean $\langle Q_L^t\rangle$.

Description of the main results obtained.

1. Concerning the local times problem, we managed to obtain a very abstract expression for the rate function of the large deviation principle we were looking for. This can be achieved applying well known results from Gamma Convergence Theory (see e.g. [B02] and [M85]) combined with the classical Gärtner-Ellis method in Large Deviations. This approach has the advantage to bypass the difficulties presented by Homogenization Theory. Nevertheless, the obtained formulas look too implicit and don't suggest a clear solution to the problem. It is necessary, therefore, to combine a coarse graining of the space with the harmonic embedding technique.
2. The second project is the one in which we obtained the most interesting results. We were able to prove a Central Limit Theorem for the Dirichlet Energy Q_L^t. The idea of the proof goes as follow: we first rewrite the quantity $Q_L^t - \langle Q_L^t\rangle$ as the sum of increments of the Doob Martingale $M_k = \mathbb{E}[Q_L^t|\mathcal{F}_k]$, where $(\mathcal{F}_i)_{i=1}^{|\Lambda_L|}$ is a carefully chosen filtration and $|\Lambda_L|$ indicates the number of edges that contribute to Q_L^t. Then the classical Central Limit Theorem for martingale arrays (note that (M_k) still depends on L) implies the required CLT provided that we can show the convergence of the square of the increments of our martingale. These increments can be expressed as the integral of the derivative of Q_L^t, and the derivative is a function of the gradient of the harmonic coordinate $\Psi^L(\omega,\cdot)$ with fixed boundary conditions (see e.g. [B11]).

   The ergodic theorem for the conductances is the natural tool to use in this case, but its implementation is not straight forward. In fact, the law of $\nabla\Psi^L$ is not translation invariant, and therefore it has to be replaced with the gradient of the full harmonic embedding $\nabla\Psi(\omega,\cdot)$ (which is translation invariant thanks to the aforementioned choice of the filtration). The main task is therefore to control the difference $|\Psi^L - \Psi|$ (in the fourth moment). This can be done furtherly approximating the two objects with their perturbed version (see e.g. [GO11]); each term has then to be approximated using different analytic techniques, such as the Meyers estimates and the Calderón-Zygmund decomposition.

Future collaboration with host institution. Developments from both projects are still ongoing work. In the second case, we plan to generalize the results obtained under the assumption of lineary boundary conditions to general Dirichlet boundary conditions and possibly a mixture of Dirichlet and Neumann boundary conditions. We are confident that this represent a feasible and concrete short-term goal, since the ingredients mentioned above are essentially waht is needed.

The contacts between Berlin and Los Angeles will anyway not end here, since the collaboration with Prof. Biskup has been extremely fruitful and motivating.

Projected publications. In the immediate future we plan to write a definitive version of our work on the Central Limit Theorem for the Dirichlet Energy. We consider the result of very high standard and will try therefore to publish it in a very prestigious mathematical journal. An aknowledgment to ESF will be surely mentioned.

Other comments. I would like to express my deepest thanks to the board of ESF for the opportunity offered to me. I consider the experience absolutely successfull from both the scientific and human point of view, and it will surely affect in a positive way my future career.

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References

[B11] M. Biskup, Recent progress on the Random Conductance Model, *Probability Surveys* **8**, (2011), 294–373.

[B02] A. Braides, Gamma-convergence for Beginners, *Oxford Lecture Series in Mathematics and Its Applications* **22**, Oxford University Press (2002).

[GO11] A. Gloria and F. Otto, An optimal variance estimate in stochastic homogenization of discrete elliptic equations, *Ann. Probab.* **39**, no.3, 779–856 (2011).

[M85] L. Modica, Gamma convergence and stochastic homogenization, *Ennio de Giorgi Colloq.*, H. Poincaré Inst., Res. Notes Math. 125, 119–125 (1985).

[W97] J.Wehr, A lower bound on the variance of conductance in random resistor networks, *J. Statist. Phys.*, **86**, no. 5-6, 1359–1365 (1997).