# SCIENTIFIC REPORT: TRICOLOR PERCOLATION ON THE PERMUTAHEDRAL LATTICE

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## 1. Purpose of Visit

This project was carried out during a visit to ETH Zurich and University of Geneva. 3 days were spent at each institution (including work over the weekend). While the initial purpose was to discuss the project with Pierre Nolin and Hugo Duminil-Copin, the visit to ETH Zurich lead to discussions with other leading researchers in the field.

The purpose of the visit was to study a certain percolation exploration model in dimensions greater than 2. The *permutahedral lattice* is a tessellation of the $d-1$ dimensional hyper-plane orthogonal to the constant vector in $\mathbb{R}^d$. It is the Voronoi tessellation of the lattice of integer valued vectors in $\mathbb{Z}^d$ whose coordinates sum to 0 and all coordinates are the same class modulo $d$. All cells are isomorphic to the $d-1$ dimensional permutahedron $\mathcal{P}_d$, whose vertices are the permutations of the vector $(1, 2, \ldots, d)$ in $\mathbb{R}^d$.

The special property of such tessellations is that $d$ cells meet at any vertex, and $d-1$ cells at any edge, the endpoints of the edge corresponding to the choice of the $d$-th additional cell. Thus, if the cells are colored in $d-1$ colors, one may explore the interface between such colors. The above properties guaranty that multi-colored edges (those edges whose adjacent cells are colored by distinct colors) form simple infinite paths or simple polygons (closed loops).

When $d = 3$ (two dimensions) the permutahedron $\mathcal{P}_3$ has 6 vertices, and the distance between two adjacent permutations is $\sqrt{2}$. $\mathcal{P}_3$ is a regular hexagon. In this case the multi-colored edges are the collection of edges in the hexagonal lattice, that are adjacent to hexagons of different colors; that is, these are the interfaces between blue and red on $p$-percolation of the hexagons. This process has been extensively studied. For $p = 1/2$, it is well known to have the conformal-invariant process $\mathsf{SLE}_6$ as the scaling limit, satisfy Cardy's formula, and many other properties are known.

The main purpose of the visit was to focus on higher dimensions, especiall dimension 3 (*i.e.* $d = 4$). For the three dimensional case $d = 4$, the permutahedron $\mathcal{P}_4$ is also known as the truncated octohedron. The permutahedral lattice in this case is isomorphic to two copies of $\mathbb{Z}^3$: the cells are the Voronoi cells of $\mathbb{Z}^3 \biguplus (\frac{1}{2}\mathbf{1} + \mathbb{Z}^3)$.

## 2. Description of the work carried out during the visit and main results

Most of the visit was dedicated to studying multi-colored percolation on the permutahedral lattice, and especially in three dimensions where the lattice is $L = \mathbb{Z}^3 \biguplus (\frac{1}{2}\mathbf{1} + \mathbb{Z}^3)$.

Analogous to the situation in dimension 2, the most basic question to answer is the existence of a phase transition: In our case, we want to establish the existence of a set in the triangle of probabilities $\{(p_1, p_2, p_3) \ : \ p_1 + p_2 + p_3 = 1\}$ for which there a.s. exist infinite components in the tricolor exploration process above. The next basic step is to determine how many infinite components exist for any specific probability vector in the triangle.

The discussion mainly focused around the question of existence of infinite exploration paths. An argument was formulated for the following result: there exists a phase for which the probabilities of long paths do not decay faster than quadratically in the distance. Together with the appropriate generalizations of classical theorems in the theory of site percolation on $\mathbb{Z}^3$, one application is a non-trivial rigorous bound on arm events of site percolation on the above lattice $L$. This implies that the critical value for site percolation on $L$ is at most $1/3$.

We also discussed arguments to push this further and to show that the critical value for site percolation on $L$ is strictly below $1/3$. Moreover, we hope that further investigation will lead to insights regarding the existence of a super-critical phase for the exploration process.

## 3. Future collaboration

This project has many paths to progress on, and we intend to continue investigation of the three dimensional case as well as higher dimensions in the future. This includes plans to carry out mutual visits which will be planned.

Hopefully, we are within reach of proving a phase transition in three dimension, which we plan to publish once the proof is fully formulated.