Short visit by Irène Marcovici (Paris 7) to James Martin (Oxford)

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Scientific report

The visit has started on 15th July 2013 and lasted for 13 days. The purpose was to collaborate in a number of directions pertaining to interacting particle systems, probabilistic cellular automata and models of random growth.

In particular, we had planned to study:

(1) the relationship between, on the one hand, the criteria for product measure marginal distributions for one-dimensional cellular automata obtained in [3] and, on the other, the conditions under which multitype exclusion processes (e.g. [4]) or queueing systems (e.g. [5]) are exactly solvable;

(2) an approach via probabilistic cellular automata to the analysis of two-dimensional spatial games related to directed percolation, of the type studied in a simpler setting on trees in [2];

(3) problems concerning models of competition growth in two dimensions and higher, concerning first-passage analogues of the last-passage models considered in for example [1].

We have discussed several questions related to points (1) and (3). Then, we have more specifically focused on point (2), and studied a game on $\mathbb{Z}^d$, related to directed percolation. Let us define this game precisely.

Let d be an integer, $d \geq 1$, and let $(e_1, \dots, e_d)$ be the standard basis of $\mathbb{R}^d$. For a given $p \in (0,1)$. The vertices of $\mathbb{Z}^d$ are closed (meaning of forbidden access) independently with probability p. This gives a configuration on which two players play the following game.

A position of the game is a vertex $x \in \mathbb{Z}^d$. From the starting position $(0, 0, \dots, 0)$, the two players play alternatively, according to the same rules. From position x, a player can move to any of the open vertices among $x + e_i, 1 \leq i \leq d$. The first player who cannot move loses the game (and the other player thus wins).

A position is said winning if a player in that position has a winning strategy (he can make the other player lose in finite time). A position is losing if a player in that position cannot move or can only move to a winning position. Positions that are neither winning nor losing correspond to draws.

We have observed that the status of the positions (win, lose, draw) could be obtained by iterating a probabilistic cellular automaton (PCA), and that the existence of draws with a positive probability was equivalent to the non-ergodicity of that PCA (we say that a PCA is ergodic if it has a unique invariant measure and if for any initial measure, the iterates of the PCA converge to that measure).

The PCA involved is closely related to the hard-core lattice gas model on two copies of the lattice $\mathbb{Z}^{d-1}$. Precisely, reversible measures for that PCA are exactly Gibbs measures for the hard-core model with an activity $\lambda(p)$ that is an explicit function of the parameter p.

For sufficiently large values of p (in particular, when there are no infinite directed paths on the configuration on which the game is played), there are no draws with probability one, and the distribution of the status of the positions can be described through the unique Gibbs measure of the hard-core model of activity $\lambda(p)$.

In contrast, when p goes to 0, the activity $\lambda(p)$ goes to infinity so that for $d \geq 3$, there is phase transition for the hard-core model (in particular, for $d = 3$, the model involved is the hard-core lattice gas model on the honeycomb lattice). This means that for p small enough, the PCA is not ergodic and that draws have a positive probability.

However, for $d = 2$, the existence of draws is unsettled: there is no phase transition for the corresponding hard-core model, and the unique Gibbs measure is a Markov chain that can be explicitly computed (so that for sufficiently large values of p, we know exactly the distribution of losing and winning positions). But for small values of p, the PCA could be non-ergodic, even if it has a unique reversible measure.

Through the study of this game, our work has highlighted non-trivial connections between directed percolation, probabilistic cellular automata, and the hard-core lattice gas model. We have been able to describe the solutions of the game for large values of p and also to prove the existence of draws for small values of p, when $d \geq 3$. The study of the case $d = 2$ and p small is still in progress and could lead to interesting developments.

In the future, we would also be interested in studying more in detail the questions related to points (1) and (3) mentionned above.

We are very grateful that is has been possible for RGLIS to support this collaboration. It has laid the ground for a more extended visit (several months) by Irène Marcovici to Oxford during the first part of 2014, for which we have obtained a (partial) funding from the Fondation Sciences Mathématiques de Paris.

[1] P. A. Ferrari, P. Gonçalves, and J. B. Martin. Collision probabilities in the rarefaction fan of asymmetric exclusion processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 45(4):1048–1064, 2009.

[2] A. Holroyd and J. Martin. Trapping and escape games on trees. In preparation, 2013.

[3] J. Mairesse and I. Marcovici. Probabilistic cellular automata and random fields with i.i.d. directions. *Annales de l'Institut Henri Poincaré*, to appear, 2013.

[4] J. Martin and P. Schmidt. Multi-type TASEP in discrete time. *ALEA Lat. Am. J. Probab. Math. Stat.*, 8:303–333, 2011.

[5] J. B. Martin. Batch queues, reversibility and first-passage percolation. *Queueing Syst.*, 62(4):411–427, 2009.