Exchange Visit Grant: Final Scientific Report

(Interactions of Low–Dimensional Topology and Geometry with Mathematical Physics)

Resurgent Analysis, Random Matrices and Gromov–Witten Invariants

1. General Information

Exchange Grant Number: ITGP 3685

Project Title: Resurgent Analysis, Random Matrices and Gromov–Witten Invariants

Start of Visit: April 1, 2012 **End of Visit:** June 30, 2012

Visitor: Ricardo Schiappa <schiappa@math.ist.utl.pt>

Host: Marcos Mariño <marcos.marino@unige.ch>

2. Purpose of the Visit

The main goal of the present exchange visit was to develop recent techniques in the analysis of random matrix models, namely, to develop the resurgent analysis of random matrix integrals and their applications in the evaluation of Gromov–Witten invariants. In particular, one of the main aims was to allow visitor and host to engage in long–term scientific discussions and, in this way, jointly address fundamental problems in this line of work.

Let us briefly recall the main scientific context of this project, following the initial proposal submitted to the ESF. Given a Calabi–Yau manifold X, the free energy of the topological A–model on X generates its Gromov–Witten invariants $N_{g,\beta}$. If the local Calabi–Yau geometry X is toric, its mirror is a certain fibration over a non–compact Riemann surface and this surface may also be understood as the spectral curve associated to the saddle–point of a specific random matrix integral. The free energy of this random–matrix partition–function may then be computed at large N, where it becomes an asymptotic expansion which solely depends on spectral geometry data. This set–up provides for a very concrete approach into the calculation of Gromov–Witten invariants. But the aforementioned asymptotic expansion only encodes perturbative data. Can one go beyond perturbation theory?

In this way it is first important to notice that, at fixed genus g, the $N_{g,\beta}$ invariants also arise from a perturbative calculation yielding a power series associated to the degree β. This is expansion is well–behaved with finite non–zero convergence radius. However, when one addresses the growth of Gromov–Witten invariants with genus, g, the story is very different: in this case general arguments in random matrix theory yield the asymptotic growth $N_{g,\beta} \sim (2g)!$. Notice that, in particular, this immediately implies that the A–model free energy is an asymptotic expansion with zero radius of convergence. As such, going beyond perturbation theory requires making sense of this asymptotic expansion.

The purpose of the visit was to investigate the use of transseries solutions in the construction of the nonperturbative structure of the aforementioned random matrix integrals. Transseries solutions consider all saddle–points of a given (matrix) integral, including both the "standard" asymptotic large N expansion, as well as all possible (nonperturbative) exponential corrections to this expansion. These exponential contributions are the key ingredient which allows us to go beyond the (perturbative) large N topological genus expansion. In this context, one of the main points we wanted to address was understanding from first principles the existence of exponential corrections of the form $\exp(+N)$, suppressed in the *negative* half–plane. In fact, these corrections do not have a first principle interpretation in terms of either a Riemann–Hilbert problem or a D–brane set–up and, as such, lead to Stokes constants which can not be computed analytically. Although much progress was made in this front (as described below) *fully* understanding what is the first principle derivation of these "generalized" instanton sectors is one open problem to be addressed in future research visits.

3. Description of the Work Carried Out

As planned, we have considered the full set of perturbative and nonperturbative corrections embodied in the transseries solution. As mentioned, this complete set of nonperturbative corrections includes previously unnoticed saddle–points of the corresponding matrix integrals, and we have made progress in trying to understand where do these corrections arise from. This was done by considering a simple non–trivial random matrix integral, the so–called quartic matrix model, and understanding its full phase structure. Because the transseries solution is constructed around a reference saddle–point, it is important to understand how it includes all other saddles and all other phases. We have found Stokes phases, anti–Stokes phases, as well as a new trivalent–tree phase, and we expect that the "generalized" instanton sectors are encoding information about all these phases. Understanding this structure is a first step in order to uncover the construction of new enumerative invariants of Calabi–Yau threefolds, as computed by topological partition functions, and which may be associated to the "generalized" instanton sectors (notice that the Calabi–Yau geometry associated to the quartic matrix model is particularly simple and, as such, a good first example to consider).

We have developed methods of resurgent analysis and its associated transseries solutions. We have extended these techniques, which were previously applied within the contexts of the Painlevé I equation and the one–cut phase of the quartic matrix model, to the contexts of the Painlevé II equation and the two–cuts phase of the quartic matrix model. We have further addressed the anti–Stokes three–cuts phase of the quartic matrix model. Resurgent solutions are Borel resummed in angular sectors of the complex plane, and distinct angular sectors are connected across Stokes' lines. In matrix models Stokes and anti–Stokes lines get generalized to Stokes and anti–Stokes *regions* and we have addressed these within the quartic matrix model. For Painlevé II and for the two–cuts phase, we have computed terms in the (possibly) infinite sequence of analytic invariants, the Stokes constants, which further encode the non–trivial information associated to the asymptotic expansions around each saddle–point.

We have further analyzed when are random matrix integrals resurgent, and how generic are the previously unnoticed saddle–points to these matrix integrals (beyond the D–brane sector). In particular, we now understand a bit better their role in the full transseries solution. Making the bridge back to the topological A–model, these results further support the idea that the new "generalized" instanton sectors may be associated to new Gromov–Witten invariants of Calabi–Yau threefolds. In this case, the full set of resurgent, enumerative Gromov–Witten invariants for a given Calabi–Yau threefold should be labeled by a set of integers $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, as $N^{(\mathbf{n})}_{g,\beta}$, associated to the k–parameters transseries solution of the dual random matrix integral. Standard Gromov–Witten invariants correspond to $\mathbf{n} = (0, \ldots, 0)$ which is the perturbative series. But their asymptotic growth, together with the expected resurgent properties of their random–matrix integral representation, point in the direction that many others should exist; as generalized Gromov–Witten invariants $N_{g,\beta}^{(\mathbf{n})}$, associated to the "generalized" instanton sectors.

4. Description of the Main Results

Following the above description we may now specify our main results, and how we have moved forward in many of our initial goals. As mentioned above, using the quartic matrix model as a prototypical example, we have fully explored its phase diagram. There are two geometrical phases, the so–called one–cut and two–cuts phases, where the saddle geometry corresponds, respectively, to a Riemann surface of genus 0 and to a surface of genus 1. Both these phases have large N duals, given by topologically distinct Calabi–Yau geometries with their respective set of Gromov–Witten invariants. This is an important point as, in particular, these Calabi–Yau geometries have *distinct* (classical) enumerative content, while they arise from the *same* random matrix integral. Further, there are non–geometrical phases (without a stringy large N dual), where many holographically dual geometries contribute simultaneously leading to the collapse of the large N 't Hooft expansion. In these "anti–Stokes" phases, the free energy is controlled by a (generalized) theta function with oscillatory behavior in N. These phases also serve as phase boundaries (they are actually regions) in–between the aforementioned one and two–cuts geometrical phases. In addition, we have also found hints of a new phase, dominated by trivalent tree–like branch–cuts, which, however, we have yet to understand from an enumerative point–of–view (it is an open question to understand if this new phase encodes and explains the appearance of "generalized" Gromov–Witten invariants). Furthermore, using data from the two–parameters transseries solution to the quartic matrix model, it was possible to perform a resummation (at lowest orders in the genus expansion) of the transseries solution, which gave origin to a (generalized) theta function. While this allows us to understand how the free energy behaves in different regions of the phase diagram, it may also open the door to a future semi–classical understanding of the new nonperturbative sectors, much in the same spirit as we presently understand the instanton sectors via eigenvalue tunneling.

In this way, the exchange visit was clearly most proficuous: we have gained much insight into the asymptotic properties of the topological free energy (albeit in specific Calabi–Yau geometries) and gained further understanding into the nature of the new resurgent sectors, fulfilling our goals. However, as natural in all scientific endeavours, many new questions were also open, paving the road for future research. As mentioned above, we have seen that different geometries arise from the same matrix model, suggesting relations between the Gromov–Witten invariants of both Calabi–Yau geometries. From the matrix model viewpoint these distinct backgrounds are related via the Stokes automorphism, further yielding a relation between two (apparently) distinct sets of Stokes constants. We are now starting to understand how this Stokes automorphism may be understood purely in the Calabi–Yau language. Furthermore, as also mentioned above, we have found new phases dominated by trivalent tree–like branch–cuts which we have not managed to understand from an enumerative point–of–view (even from a matrix model perspective, recent work on these structures has been purely numerical). Uncovering what these structures really encode is a very interesting question. Finally, one of our initial questions which still remains open concerns the enumerative interpretation of the new resurgent Gromov–Witten invariants. "What exactly are they counting?", is a fascinating question which we must leave for future research, hopefully during future exchange visits.

5. Future Collaboration with the Host

The present visit and its subsequent research activities have made it possible for both host and visitor to reignite what was their long history of research collaboration and scientific exchange. As described above, while many goals were achieved, at the same time some have still to be fully understood and many new research questions were also put forward. In this way many new scientific projects between host and visitor can be envisaged for the near future. In particular, in order to proceed with the present line of research, more visits of scientific exchange will be required (with either the present applicant returning to Geneva in order to visit the host, or with the present host traveling to Lisbon instead) in order to fully pursue all scientific research lines which started with the present collaboration. In this way, and in order to pursue our combined long–term research plan, we are already planning a new visit of the applicant to the University of Geneva within the next 12 months so that, in this way, we may continue developing a very profitable research collaboration.

6. Projected Publications

Two publications are presently being written and, as soon as they become available, copies will be forwarded to the ESF. In one publication we explore complimentary phases of the quartic matrix model, in particular we address the two–cuts phase of this model and study its resurgent properties. This allows us to understand a distinct Stokes phase of this model with distinct enumerative (Gromov–Witten) content. It also yields insight into the resurgent structure of the Painlevé II equation. In another paper we show how the aforementioned phase, together with the usual one–cut phase we have previously address, combine together into a phase diagram where one can explore Stokes phenomena (and where new phases also make an appearance). We explore anti–Stokes phase transitions and, by linking distinct Stokes phases via the Stokes automorphism, we show how distinct saddles relate to each other. This may lead to relation between the enumerative content of the two distinct backgrounds. We further resum the transseries asymptotic expansion in order to show how it yields generalized theta functions, and explore the role they play in understanding both the anti–Stokes phase and the trivalent tree–like phase.