# ESF/ITGP REPORT FOR A VISIT AT CRM, BARCELONA JUNE 12-26, 2012

FLORENT SCHAFFHAUSER

## Contents

This is a report on my research stay at CRM, Barcelona in between the dates June 12-26, 2012 (2 weeks), in the frame of the programme *Geometry and Quantization of Moduli Spaces*.

## 1. Purpose of the visit

The purpose of my two-week visit was twofold:

- Resume an ongoing collaboration with Professor Oscar García Prada on *Higgs bundles over real algebraic curves*.
- Participate in the conference *Geometry and quantization of moduli spaces*, to which I was invited by Professor Ignasi Mundet i Riera.

## 2. Description of the work carried out during the visit

My time in Barcelona was organized around the main two objectives described above. The conference week, in particular, was one of intense scientific exchange with several participants in the conference. The talk given by Olivier Guichard on Hitchin representations, in particular, was very related to my current interests and stimulated a new direction in my research.

As for my ongoing research project with Oscar García Prada, let me mention the following aspects. The idea is to study (principal) Higgs bundles over a real algebraic curve $(X, \sigma)$ and to obtain analogues of the Hitchin-Kobayashi and Donaldson-Corlette correspondences in this setting. While real objects in the category of Higgs bundles are straightforward to define, the interesting notion to pin down in this work is the appropriate notion of stability. By appropriate, we mean that the associated poly-stable objects should correspond to representations of the real fundamental group of the base curve.

The latter group is an object not so familiar to differential geometers, even though it might be defined in a purely topogical, very elementary way. Part of the effort during my stay at the CRM was to give introductory talks on this group (this was actually accomplished during an earlier visit, in May, where I gave a mini-course

in the *Master Class on Surface Group Representations*). This is directly related to the research project with García Prada. As a matter of fact, a good understanding of this group gives the map that one needs to show is a bijection to obtain the Donaldson-Corlette correspondence over a real algebraic curve.

The idea of our proof is to generalise the approach in an earlier paper of mine (*Real points of coarse moduli schemes of vector bundles on a real algebraic curve*, to appear in *J. Symp. Geom.*) to the Higgs context. The main result therein is a real version of Donaldson's formulation of the Narasimhan-Seshadri correspondence. This is carried out by a careful analysis of the notion of stability for real vector bundles. It should be mentioned that the proof is purely geometrical and that all the hard analysis results are contained in Donaldson's statement. The novelty stands purely in the algebro-geometric notions at stake, which are in fact quite elementary compared with the analytical aspects. The main observation is that semi-stability and poly-stability for real vector bundles coincides with semi-stability and poly-stability of the underlying holomorphic bundle, while there are real bundles which are stable in a real sense but only poly-stable in the holomorphic sense. This feature is actually necessary to guarantee the existence of real Jordan-Hölder filtrations, on the existence of which the proof of the real analogue of Donaldson's theorem is based.

## 3. Description of the main results obtained

During my stay, Oscar García Prada and I were able to complete a proof of the generalisation of the afore-mentioned results to real Higgs *vector* bundles. The analysis of stability in this case reveals the exact same phenomena for Higgs vector bundles as for ordinary vector bundles and one can then prove the analogues of the Hitchin-Kobayashi and Donaldson-Corlette correspondences by finding real analogues of the usual Kemp-Ness-like approach in each case.

As for general principal Higgs bundles, the stability notion is, as is to be expected, more intricate. In particular, it is not entirely clear at present time whether it is sufficient to consider poly-stability of the associated adjoint bundle to obtain the Hitchin-Kobayashi and Donaldson-Corlette correspondences. And even if this were the case, this would still leave the problem of finding moduli for stable objects partially unsolved.

## 4. Projected publications resulting from the visit

A joint paper with Oscar García Prada on the Hitchin-Kobayashi and Donaldson-Corlette correspondences for real Higgs bundles is in preparation. As mentioned above, some of the results that would be contained in this paper were obtained during my stay at the CRM.