

ESF Exploratory Workshop on  
**Mathematical aspects of the physics  
with non-self-adjoint operators**

Prague (Czech Republic), 30 August - 3 September 2010

Convened by:  
**David Krejcirik, Jean-Pierre Gazeau and Petr Siegl**

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**SCIENTIFIC REPORT**

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Doppler Institute for mathematical physics and applied mathematics,  
Czech Republic

# 1. Executive summary

Many physical systems can be described by partial differential equations and the latter can often be viewed as generating abstract operators between Banach spaces. A typical example is quantum mechanics where the traditional mathematical discipline for the study is the functional analysis of self-adjoint operators in a Hilbert space. In other areas of physics a more general class of operators is necessary to describe a process in Nature. In fact, already in quantum theories there are problems which require an analysis of non-self-adjoint operators, too.

From the mathematical point of view, the theory of self-adjoint operators is well understood, while the non-self-adjoint theory is still in its infancy. It is frustrating that the powerful techniques of the former, such as the spectral theorem and variational principles, are not available for the latter. Moreover, recent studies have revealed that this lack of tools is fundamental, the non-self-adjointness may lead to new and unexpected phenomena. Although there exist many interesting observations coming from physics and numerical experiments with non-self-adjoint problems, the deep theoretical understanding is still missing. We tried to use the strong motivations for rigorous studies of non-self-adjoint systems to attract more attention from the mathematical community.

The purpose of this workshop was to gather mathematicians and mathematical physicists encountering the study of non-self-adjoint operators in various contexts, with different goals and techniques. The activities are widely scattered in Europe and some research groups are even not aware of the related parallel projects. The reasons for this situation can be found in both, at first sight, different nature of the projects (pure mathematics, mathematical physics, particular branches of physics) and in the diverse developments in the fields (well established areas of mathematics and physics versus very new, rapidly and narrowly developing disciplines). We hoped to encourage more interaction among the approaches and contribute in this way to a better understanding of the non-self-adjoint theory and its application in physics. The list of participants involved leading researchers working on non-self-adjoint operators with wide-ranging motivation, the experts in mathematics and mathematical physics who would not ordinarily participate in the same research group.

The workshop was aiming at mathematical studies of non-self-adjoint problems coming from modern physics with a special emphasis put on rigorous justifications of phenomena encountered in the newly developing fields represented by PT-symmetric quantum mechanics and quantization in the de Sitter spacetime. Among other topics covered, the workshop was particularly concerned with:

- PT-symmetric quantum mechanics
- quantization in the de Sitter spacetime
- non-self-adjoint perturbation theory
- quantitative bounds on operators, pseudospectra
- complex symmetric operators
- Krein spaces
- damped wave systems

The non-self-adjoint theory is extremely diverse, it comprises a collection of advanced methods and it is difficult, if not impossible, to find a common thread. From the purely

mathematical point of view, the workshop represented an interaction of functional analysis, notably spectral theory, partial differential equations, and analytic function theory.

The workshop took place from the 30th August to the 3th September 2010, the border dates being arrival and departure days, respectively. It was held in Villa Lanna, a small conference centre of the Academy of Sciences of the Czech Republic in Prague, perfectly suitable for the interactive nature of the meeting. We believe that the one-week activity stimulated interesting scientific discussions in a friendly atmosphere created by the participants. The workshop brought together 27 researchers from 10 countries.

Let us conclude this summary by quoting E. B. Davies from the preface in his recent book *Linear operators and their spectra* (Cambridge 2007), where a significant amount of work on spectral theory of non-self-adjoint operators can be found:

*Studying non-self-adjoint operators is like being a vet rather than a doctor: one has to acquire a much wider range of knowledge, and to accept that one cannot expect to have as high a rate of success when confronted with particular cases.*

We fully endorse this opinion and hope that the outcomes of the workshop will contribute to a better understanding of the non-self-adjoint theory.

## 2. Scientific content

The workshop was intended to provide a fruitful atmosphere for discussions and to initiate new or deepen already established joint research works on themes involving the theory of non-self-adjoint operators. In the morning session of each day we had two introductory talks and one contributed talk followed by a short discussion session. Each afternoon consisted of two introductory talks, of two additional contributed talks and was concluded by an extensive discussion session. The talks covered the main topics of the workshop; the speakers were asked to present their fields of interest in a compact, comprehensive form, and to focus on open problems. The discussion sessions were dedicated to open problems; interested participants presented specific open problem on an interactive way with the other participants using blackboard. Furthermore, also the last 10 minutes of each talk was strictly reserved to questions and short discussions, in order to give the participants the first occasion to share ideas about the discussed topics. The discussions continued on a more informal level after dinner.

The scientific programme was opened with a talk by E. B. Davies on asymptotic distribution of resonances of quantum graphs. Here the fundamental (Schrödinger) equation is self-adjoint, but the method of complex scaling leads to the spectral analysis of a non-self-adjoint operator. The recent results of the speaker and his coauthors reveal an interesting fact that the Weyl law for resonances may not hold, depending on the geometric structure of the graph.

J. Sjöstrand reported on distribution of eigenvalues of non-self-adjoint Schrödinger-type operators by the methods of microlocal analysis. It was pointed out how the spectral instability can play a beneficial role when one considers small random perturbations. A large number of open problems was mentioned at the end of his talk.

The first part of the afternoon session was opened with a talk by S. R. Garcia on complex symmetric operators, highlighting the important role played by certain antilinear symmetries. The second part was dedicated to mathematical aspects of PT-symmetric quantum mechanics. Here the contributed talk by P. Siegl on open problems in PT-symmetry spontaneously led to a vivid follow-up discussion.

The second day began with a very interactive presentation (on blackboard) by K. Veselic on the limits of spectral information in non-self-adjoint problems such as the quadratic operator pencil. The ideas were further supported by the follow-up talks by X. P. Wang and E. Zuazua on dissipative Helmholtz and wave equations, respectively.

The afternoon session was dedicated to non-self-adjoint problems arising in the context of indefinite inner product spaces. It was opened with an introductory talk by H. Langer and followed by talks by Ch. Tretter, S. Kuzhel and C. Trunk. A number of open problems was presented in the discussion session.

The morning session of the third day was dedicated to the topic of the quantization of the de Sitter space. It was opened with an introductory talk by J. Bros and followed by talks by J.-P. Gazeau and H. Bergeron.

In the last afternoon we had talks by G. Bouchitté on a non-self-adjoint problem coming from the electromagnetic theory of metamaterials and by A. Laptev on Lieb-Thirring-type inequalities for eigenvalues of Schrödinger operators with complex potentials.

The scientific programme concluded with the discussion of follow-up research activities, notably in the framework of ESF.

### **3. Assessment of the results, contribution to the future direction of the field, outcome**

The workshop fulfilled our aims to bring together previously almost disconnected research groups studying non-self-adjoint problems in different contexts and with various methods. We believe that the workshop will stimulate further cooperation among these groups, which will hopefully lead to research articles published in internationally prestigious journals.

The first visible outcome of the workshop is a list of open problems formulated by the participants during or after the workshop that is gradually updated on the webpage of the event:

[www.ujf.cas.cz/ESFNSA](http://www.ujf.cas.cz/ESFNSA)

Moreover, the list of the open problems is expected to be published in Journal of Integral Equations and Operator Theory (Birkhäuser).

On the last day of the workshop when possible follow-up research activities were discussed, it was suggested by the senior participants that the established interaction of research groups should be definitely followed by other meetings of this type or even by a larger-scale conference. It became clear during the discussion that it would be desirable to enrich the potential follow-up activity by other themes, such as non-self-adjoint (random) matrices and

numerical aspects of the non-self-adjoint theory, topics which are currently very active and were under-represented in the workshop.

Checking the possible activities offered on the annual base of the ESF, it was agreed to try to apply for a conference in the ESF Research Conferences Scheme. Since the deadline for the application for conferences to take place in 2012 was shortly after the workshop, a broad discussion took place on how to on the best way to present this proposal. It turned out that in 2012 there will be a larger meeting *Operator Theory, Analysis and Mathematical Physics*, as a part of a series of conferences organized every two years by J. Janas (Poland), P. Kurasov (Sweden), A. Laptev (UK) and S. Naboko. It was decided to prepare a joint proposal, employing at the same time the tradition of the well established OTAMP meetings and the recent success of our workshop.

## 4. Final programme

### Monday, 30 August 2010

morning arrivals

18:00 welcome drink

### Tuesday, 31 August 2010

08:45 - 09:00	presentation of ESF	
09:00 - 10:00	E. B. Davies	<i>Resonances of quantum graphs</i>
10:00 - 10:45	H. Neidhardt	<i>On perturbation determinants for non-selfadjoint operators</i>
10:45 - 11:15	coffee & tea break	
11:15 - 12:15	J. Sjostrand	<i>Distribution of eigenvalues of non-self-adjoint differential operators and spectral instability</i>
12:15 - 12:30	discussion	
12:30 - 14:00	lunch	
14:00 - 15:00	S. Garcia	<i>Complex symmetric operators: an overview</i>
15:00 - 15:45	L. Boulton	<i>A non-orthogonal spectral method in one dimension</i>
15:45 - 16:15	coffee & tea break	
16:15 - 17:15	M. Znojil	<i>PT-symmetric Hamiltonians: Non-self-adjoint or self-adjoint?</i>
17:15 - 18:00	P. Siegl	<i>Open problems in PT-symmetry</i>
18:00 - 19:00	discussion	
19:00 - 20:30	dinner	

### Wednesday, 1 September 2010

09:00 - 10:00	K. Veselic	<i>How reliable is non-selfadjoint spectrum?</i>
10:00 - 11:00	X. P. Wang	<i>High frequency analysis of the dissipative Helmholtz equation</i>
11:00 - 11:30	coffee & tea break	
11:30 - 12:15	E. Zuazua	<i>Dissipative wave equations: asymptotic behavior and numerics</i>
12:15 - 12:30	discussion	
12:30 - 14:00	lunch	
14:00 - 15:00	H. Langer	<i>Spectral problems in indefinite inner product spaces</i>
15:00 - 15:45	Ch. Tretter	<i>Spectral inclusion for non-selfadjoint operators</i>
15:45 - 16:15	coffee & tea break	

16:15 - 17:00	S. Kuzhel	<i>On a class of self-adjoint operators in Krein space with empty resolvent set</i>
17:00 - 17:45	C. Trunk	<i>On domains of PT symmetric operators related to <math>-y''(x) + (-1)^n x^{2n} y(x)</math></i>
17:45 - 19:00	discussion	
19:00 - 22:00	festive dinner	

### Thursday, 2 September 2010

09:00 - 10:00	J. Bros	<i>Solving an indefinite-metric problem: Scalar tachyons in the de Sitter universe</i>
10:00 - 10:45	J.-P. Gazeau	<i>The de Sitter group representations: old results and new questions</i>
10:45 - 11:15	coffee & tea break	
11:15 - 12:00	H. Bergeron	<i>Coherent states based on non-self-adjoint operators: From the harmonic oscillator to any one dimensional Hamiltonian</i>
12:00 - 12:30	discussion	
12:30 - 14:00	lunch	
14:00 - 15:00	G. Bouchitte	<i>Transmission between media with opposite sign dielectric constants and anomalous resonances</i>
15:00 - 15:45	discussion	
15:45 - 16:15	coffee & tea break	
16:15 - 17:15	A. Laptev	<i>Eigenvalue estimates for Schrodinger operators with complex potentials</i>
17:15 - 17:45	discussion	
17:45 - 19:00		<i>planning of follow-up research activities</i>
19:00 - 20:30	dinner	

### Friday, 3 September 2010

morning                      departures

## 5. Final list of participants

- Hervé BERGERON**  
Institut des Sciences Moléculaires d'Orsay  
Université Paris-Sud, France
- Guy BOUCHITTE**  
Laboratoire IMATH, U.F.R. des Sciences et Techniques  
Université du Sud-Toulon-Var, France
- Lyonell BOULTON**  
Department of Mathematics  
Heriot-Watt University Edinburgh, United Kingdom
- Jacques BROS**  
Service de Physique theorique - CEA  
Saclay, France
- Monique COMBESURE**  
Institut de Physique Nucléaire de Lyon  
France

6. **E. Brian DAVIES**  
Department of Mathematics,  
King's College London, United Kingdom
7. **Pavel EXNER**  
Department of Theoretical Physics  
Nuclear Physics Institute ASCR, Czech Republic
8. **Tamas FULOP**  
KFKI Research institute for particle and nuclear physics of the Hungarian Academy of Sciences  
Budapest, Hungary
9. **Stephan R. GARCIA**  
Department of Mathematics  
Pomona College, Claremont, USA
10. **Jean-Pierre GAZEAU** (co-convenor)  
Laboratoire Astroparticule et Cosmologie,  
Université Paris Diderot – Paris 7, France
11. **Céline GIANESELLO**  
Department of Theoretical Physics  
Nuclear Physics Institute ASCR, Czech Republic
12. **Manuel GONZALEZ**  
Department of Mathematics, Universidad de Cantabria  
Santander, Spain
13. **Uwe GUNTHER**  
Technical University Kaiserslautern,  
Germany
14. **David KREJCIRIK** (convenor)  
Department of Theoretical Physics  
Nuclear Physics Institute ASCR, Czech Republic
15. **Sergey KUZHEL**  
Institute of Mathematics, Ukrainian National Academy of Sciences,  
Kiev, Ukraine
16. **Heinz LANGER**  
Institute of Analysis and Scientific Computing  
Vienna University of Technology, Austria
17. **Ari LAPTEV**  
Department of Mathematics  
Imperial College London, United Kingdom
18. **Marco MARLETTA**  
Cardiff School of Mathematics, United Kingdom
19. **Hagen NEIDHARDT**  
Weierstrass Institute for Applied Analysis and Stochastics  
Berlin, Germany
20. **Petr SIEGL** (co-convenor)  
Department of Theoretical Physics  
Nuclear Physics Institute ASCR, Czech Republic

**21. Johannes SJOSTRAND**

Department of Theoretical Physics  
Université de Bourgogne, Dijon, France

**22. Christiane TRETTER**

Mathematisches Institut  
Universitaet Bern, Switzerland

**23. Carsten TRUNK**

Institut für Mathematik  
TU Ilmenau, Germany

**24. Kresimir VESELIC**

Fakultaet fuer Mathematik und Informatik  
Fernuniversitaet in Hagen, Germany

**25. Xue Ping WANG**

Laboratoire de Mathématiques Jean Leray, UMR CNRS 6629  
Université de Nantes, France

**26. Miloslav ZNOJIL**

Department of Theoretical Physics  
Nuclear Physics Institute ASCR, Czech Republic

**27. Enrique ZUAZUA**

Basque Center for Applied Mathematics,  
Bilbao, Spain

**6. Statistical information on participants** (age bracket, countries of origin, M/F repartition, etc.) The statistics to be provided under section 6 can also include repartition by scientific specialty if relevant.

*Countries of origin:*

Austria	1
Czech Republic	5
France	7
Germany	4
Hungary	1
Spain	2
Switzerland	1
UK	4
Ukraine	1
USA	1

*Gender:*

Female	3
Male	24

*Age bracket:*

20-30	2
30-40	5
40-50	4
50-60	7
60-...	9