EUROPEAN SCIENCE FOUNDATION
SETTING SCIENCE AGENDAS FOR EUROPE

Exploratory Workshop Scheme

Standing Committee for Physical and Engineering Sciences (PESC)

ESF Exploratory Workshop on

Current Problems in Differential Calculus over Commutative Algebras, Secondary Calculus, and Solution Singularities of Nonlinear PDEs

Vietri sul Mare (Italy), June 13-16, 2011

Convened by:

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SCIENTIFIC REPORT

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1. Executive summary

The workshop took place form June 13 to June 16, 2011 (half of both the border days were spent in scientific activities) in Hotel Lloyd's Baia, Vietri sul Mare (SA), Italy. Participation numbered 22 participants (including the ESF Rapporteur) from 10 different EU countries and 2 extra EU countries. The confortable location favoured a friendly atmosphere and a number of informal scientific discussions during the breaks took place. The workshop definitely stimulated the interaction of scientists from different areas with a likely profit for the field.

The scope of the workshop was to discuss recent developments in the modern geometrical theory of nonlinear partial differential equations (NPDE) and their solution singularities with a special emphasis on its interactions with mathematical and theoretical physics. In this respect, differential calculus over commutative algebras and secondary calculus are natural languages for the three wide fields of *differential geometry*, *partial differential equations* and *theoretical physics*.

In fact, modern geometrical theories of nonlinear PDEs have already manifested their power in mathematical and theoretical physics, mechanics and control theory. However, their full potential is still to be discovered and developed. This is mostly due to their highly interdisciplinary character, a nonstandard and yet natural synthesis of various elements coming from geometry of jet spaces, differential and algebraic geometry, commutative and homological algebra, algebraic and differential topology, differential calculus over commutative (graded, super) algebras, theory of integrable systems. Several internationally recognized researchers both in related theoretical and applicative areas are scattered throughout Europe. The workshop aimed to convene a distinguished group of them to build up interfaces between different expertises.

The three main themes discussed in the workshop agenda:

Differential calculus over (graded) commutative algebras. Now is becoming clear that a natural language of differential geometry is differential calculus over algebras, e.g., smooth function algebras on manifolds. It is indispensible when dealing with diffieties, a kind of infinite-dimensional manifolds that is at the basis of the modern geometric theory of NPDEs. The workshop put together both differential geometers and experts in various algebraic approaches to geometry with the aim of establishing a common efficient language in a perspective of extending methods of differential geometry to commutative algebra (in particular, to manifolds with singularities) and other recently emerged related areas and developing new efficient computational techniques.

Singularities of multi-valued solutions of PDEs. Recently, it was recognized that eiconal-type equations constitute just a part of the system of subsidiary equations describing dynamic of singularities of solutions of a PDE. This put the quantization problem in Quantum Mechanics, and such problems as that of turbulence or fissure, in the framework of the theory of solution singularities of NPDEs. The workshop put together experts in the geometric theory of solution singularities of PDEs on one side and experts in i) quantum mechanics and ii) dynamical systems on the other side in order i) to discuss new promising perspectives of implementation of new methods of geometrical theory of NPDEs to quantum mechanics and ii) to elaborate solid foundations of dynamics of singularities for current need of mechanics of continua.

Secondary Calculus. Secondary Calculus is a proper formalization in algebro-geometric terms of the idea of a differential calculus on the space of solutions of a NPDE. One of its main problem, the so-called secondarization problem, is to construct and study analogues of objects of the traditional differential calculus and differential geometry which, from the point of view of possible applications to field theory, is analogous to the quantization problem. Interpreting higher symmetries of NPDE, conservation laws, recursion operators, etc, as homogeneous components of suitable secondarized objects, was the first success of Secondary Calculus from both theoretical and practical viewpoints. The workshop put together experts of Secondary Calculus on one side and experts of i) NPDEs, ii) quantum field theory, iii) categorical approach to functional differential calculus with the aim of establishing i) a common efficient language for the study of NPDEs, ii) the contribution that Secondary Calculus may give in quantum field theory, iii) an efficient categorical language for Secondary Calculus.

By summing up, the workshop convened leading experts in Differential Geometry, NPDEs, and Mathematical/Theoretical Physics. The interconnections between such fields are widely recognized. However, a systematic interaction among experts is still missing. The main aim of the workshop was to establish a common platform for setting and studying old and new problems in all these three fields and their interfaces, with a special emphasis on the themes in the title.

2. Scientific content of the event

The full program, with downloadable abstracts and transparencies of (most of) the talks, can be found at the e-address:

http://www.levi-civita.org/Activities/workshops/ESF-EW-2011

13 June (Afternoon)
Alexandre VINOGRADOV (*Welcome address*) remainded the main objectives of the conference.
Aiyalam BALAHANDRAN (*Causality and Statistics on the Moyal Plane*) discussed non-commutative algebraic violations of causality at the Plank scale and their eventual observable effects.
Giuseppe MARMO (*Quantum Information and Related Inequalities in the Geometrical Formulation of Quantum Mechanics*) discussed geometric methods in both statistical theories and quantum mechanics with a special enphasis on the role of Käler structures.
Alexandre VINOGRADOV (*Solution Singularities of Non-Linear PDEs*) discussed the geometric theory of singularities of multivalued solutions of NPDEs with an emphasis on open problems: classification of type-Σ singularities, classical to quantum transition via reconstruction theorems.
Discussions. See next section.

14 June (Morning)
Janusz GRABOWSKI (*Dirac Algebroids*) presented the novel concepts of a Dirac and Dirac-Lie Algebroid and its main geometric and algebraic properties.
Katarzyna GRABOWSKA (*Mechanics on Dirac Algebroids*) discussed the application of Dirac Algebroids to the theory of constrained Lagrangian and Hamiltonian systems and their reduction.
Hong Van LE (*Geometric Structures Associated with a Simple Cartan 3-form*) discussed the geometry of simple Cartan 3-forms with a special enphasis on their link to multisymplectic geometry and de Rham cohomology.
Silvia VILARIÑO FERNÁNDEZ (*Polysymplectic Structures and Marsden-Weinstein Reduction for Field Theories*) discussed the Polysymplectic approach to classical field theory and an polysymplectic analogue of the Marsden-Weinstein reduction procedure.

14 June (Afternoon)

Norbert PONCIN (*Loday algebroids and their super-geometric interpretation*) presented the concept of Loday algebroid and discussed the possibility of describing Loday algebraic structures in terms of homological vector fields on suitable supermanifolds.

Vyacheslav KALNITSKY (*Tangent Space of a Commutative Algebra*) reviewed the commutative algebraic approach to the Poisson geometry of the cotangent bundle and applied it to the description of the cotangent bundle of a simple singular manifold: the cross.

Elizaveta VISHNYAKOVA (*Locally Free Sheaves on Complex Supermanifolds*) presented a classification theorem for a special class of locally free sheaves on a complex supermanifolds and some corollaries.

Discussions. See next section.

15 June (Morning)

Jerzy KIJOWSKI (Can we solve Schroedinger equation via purely geometric methods?) presented an original point of view on geometric quantization: the Schrödinger equation describes the change of the geometric quantization picture under the change of the polarization along the classical Hamiltonian flow.

M. Eugenia ROSADO MARÍA (*Examples of PDEs All Whose Points Are Characteristic*) discussed the formal integrability of the Euler-Lagrange equations corresponding to diffeomorphism invariant Lagrangian on the bundle of frames proving that they are formally integrable even if they cannot be put in Cauchy-Kovalewska form.

Evgeny FERAPONTOV (*On the Integrability of Symplectic Monge-Ampère Equations*) introduced a notion of integrability for the class of symplectic Monge-Ampère equations (SMAE) and classified integrable SMAE in four dimensions. He put a special emphasis on the geometric formulation of the problem and on open questions.

Michal MARVAN (*Diffieties and The Minimal Set of Compatibility Conditions*) discussed how to get internal coordinates on a diffiety associated on a system of PDEs using recent developments in the Riquier theory of compatibility conditions.

Johan VAN DE LEUR (*Integrable Systems and Multiple Orthogonal Polynomials of Mixed Type*) showed that multiple orthogonal polynomial are connected to the multi-component KP hierarchy and presented a recursion relation on them.

15 June (Afternoon)

Sergey IGONIN (Lie Algebras and Algebraic Curves Responsible for Baecklund Transformations of *PDEs*) presented a Secondary Calculus analogue of the Fundamental group, i.e., the fundamental Lie algebra of a PDE, and its application to the geometric theory of PDEs and their coverings.

Frédéric PAUGAM (*Homotopical Geometry and the BV Formalism*) describe a categorical approach to Secondary Calculus with a special emphasis on its role in the BV formalism for the quantization of gauge theories.

Hans-Christian HERBIG (*On Deformations of Singular Poisson Algebras*) discussed the deformations of Poisson algebras resulting from the reduction of the fiber of a momentum map over a non regular value.

16 June (Morning)

Giovanni MORENO (*The Geometry of Jet Spaces of Pairs of Manifolds*) discussed, within Secondary Calculus, a new geometric setting for the intrinsic description of boundary value / initial data problems for PDEs.

Luca VITAGLIANO (*Hamilton-Jacobi Diffieties*) presented, within Secondary Calculus, a new understanding of the standard Hamilton-Jacobi problem in Hamiltonian mechanics, and a generalization to generic systems of PDEs.

3. Assessment of the results, contribution to the future direction of the field, outcome

The workshop brought together researchers from rather disconnected groups working in differential geometry and geometrical theory of NPDEs and/or Theoretical and Mathematical Physics. participants are very optimistic that the meeting will promote further cooperation among these researchers with a general profit for all involved scientific areas.

In fact, the workshop resulted in a general agreement, emerged during the discussions, on the central role of Geometry, in particular Differential Geometry, in Mathematics and Physics, especially concerning the theory of NPDEs on one side, and Mathematical and Theoretical Physics on the other side. Namely, participants agreed on the fact that Geometry is an adequate universal language for building fundamental mathematical and physical theories, ranging from the theory of NPDEs and dynamical systems, to theoretical mechanics, mechanics of continuum media, quantum mechanics and quantum field theory. In particular, they agreed that research should aim to the development of such a geometrical language. This agreement is of a particular importance in view of the fact that the list of participants counted distinguished researchers in both Mathematics and Physics. It can be stated that the workshop contributed to the creation of a European community of researchers from different areas and with different expertises and nonetheless aiming at the common scope of creating a new geometric vocabulary, and methods based on it, as a pre-eminent tool in affording the challenges of contemporary Mathematics and Physics. In this respect, the synergy between such different expertises seems to have a special importance.

More specifically, a number of concrete scientific objectives was identified during the discussions. Namely,

-Regarding **Differential Calculus over Commutative Algebras**: of a special interest would be to systematically develop
i) differential calculus over algebras of smooth functions on manifolds with singularities as a method to extend classical geometric tools of analytical mechanics to a much broader area of applications.
ii) differential calculus over graded algebras as a new powerful tool in modelling a wide range of physical phenomena.
iii) the theory of algebroids (Lie, Courant, Poisson, Dirac, Loday, etc) and their strong homotopy counterparts naturally emerging as algebraic-geometric structures in different contexts in both mathematics and physics aiming to contribute in better understanding of the underlying mathematical structures of many theories ranging from deformation theory, to theoretical mechanics of contrained systems and field theory.
iv) a systematical study of the relationships between commutative algebra and non-commutative geometry in the context of their applications to quantum mechanics and quantum field theory.

- Regarding **Secondary Calculus**: of a special interest would be to develop
i) new cohomological tools and techniques based on formal integrability theory for the efficient computations of the C-spectral sequences and similar objects in geometrical theory of NPDEs.
i) the theory of the fundamental Lie algebra of a system of NPDEs which is an analogue of the fundamental group of a manifold in geometricval theory of NPDEs and already proved to have a very important role in the theory of integrable systems. However, a general satisfactory definition of this algebra is still missing. Finding it would be a breakthrough point in geometrical understanding of NPDEs.

ii) a geometrical theory of initial data / boundary condition problems for PDEs as a part of Secondary Calculus. This would noteworthy enrich methods of Secondary Calculus not only in a wide range of problems in NPDEs and Theoretical Physics involving initial data / boundary conditions, but also in the calculus of variations (generalized transversality conditions), "relative" conservation laws, etc.

- Regarding the **Geometric Theory of PDEs and their Solution Singularities**: of a special interest would be
i) to create a common geometrical platform unifying concrete methods previously developed in the study of particular classes of PDEs (equations of hydrodinamical type, Monge-Ampere equations, etc) in the perspective to generalize them onto wider classes of NPDEs by using tools of the modern geometry of PDEs.
ii) to study the relation between the equations of motion in continuum mechanics and the singularities of their solutions. In many cases a PDE can be reconstructed from the dynamics of singularities of its solutions. In particular, this phenomenon allows one to "recostruct the field equations from the motion of the field sources". On the other hand the reconstruction procedure is analogous to the quantization procedure. The systematic study of solution singularities of non-linear PDEs would very plausibly shed new light on the problem of mathematical modelling of turbulence in continuum media and similar phenomena in field theory and the quantization problem as well.
Finally, the participants agreed that it would be very important not to disperse their scientific efforts. Moreover, they emphasized that to successfully face above listed problems additional human resourses are needed. In order to achieve this goal, it would be very useful to constitute a research network at the European scale. The participants intend to plan a follow up meeting with the aim of discussing 1) details of a concrete common research projects on the above mentioned themes, 2) the possibility of applying for a Marie Curie action to finance such project. Concluding, it has been decided that the convenors will submit to the ESF an application for financial support of such follow up meeting.

4. Final programme

Monday, 13 June 2011

morning	*Arrival*
13.00	*Welcome Lunch*
15.15-15.30	**Welcome by Convenor** **Alexandre VINOGRADOV**
15.30-15.50	**Presentation of the European Science Foundation (ESF)** **Manuel DE LEON**
15.50-19.30	**Afternoon Session**
15.50-16.25	**"Causality and Statistics on the Moyal Plane"** **Aiyalam BALAHANDRAN**
16.25-17.00	**"Quantum Information and Related Inequalities in the Geometrical Formulation of Quantum Mechanics"** **Giuseppe MARMO**
17.00-17.30	*Coffee Break*
17.30-18.05	**"Solution Singularities of Nonlinear PDEs"** **Alexandre VINOGRADOV**
18.05-19.30	**Discussion**
20.00	*Dinner*

Tuesday, 14 June 2011

09.05-12.30	**Morning Session**
09.05-09.40	**"Dirac Algebroids"** **Janusz GRABOWSKI**
09.40-10.15	**"Mechanics on Dirac Algebroids"** **Katarzyna GRABOWSKA**
10.15-10.45	*Coffee Break*
10.45-11.20	**"Geometric Structures Associated with a Simple Cartan 3-form"** **Hong Van LE**
11.20-11.55	**"Polysymplectic Structures and Marsden-Weinstein Reduction for Field Theories"** **Silvia VILARIÑO FERNÁNDEZ**
13.00	*Lunch*
15.15-19.30	**Afternoon Session**
15.15-15.50	**"Loday algebroids and their super-geometric interpretation"** **Norbert PONCIN**
15.50-16.25	**"Tangent Space of a Commutative Algebra"** **Vyacheslav KALNITSKY**
16.25-17.00	**"Locally Free Sheaves on Complex Supermanifolds"** **Elizaveta VISHNYAKOVA**
17.00-17.30	*Coffe Break*
17.30-19.30	**Discussion**
20.00	*Dinner*

Wednesday, 15 June 2011

09.05-12.30	**Morning Session: Secondary Calculus I**
09.05-09.40	**"Can we solve Schroedinger equation via purely geometric methods?"** **Jerzy KIJOWSKI**
09.40-10.15	**"Examples of PDEs All Whose Points Are Characteristic"** **M. Eugenia ROSADO MARÍA**
10.15-10.45	*Coffee Break*
10.45-11.20	**"On the Integrability of Symplectic Monge-Ampère Equations"** **Evgeny FERAPONTOV**
11.20-11.55	**"Diffieties and The Minimal Set of Compatibility Conditions"** **Michal MARVAN**
11.55-12.30	**"Integrable Systems and Multiple Orthogonal Polynomials of Mixed Type"** **Johan VAN DE LEUR**
13.00	*Lunch*
15.15-19.30	**Secondary Calculus II**
15.15-15.50	**"Lie Algebras and Algebraic Curves Responsible for Baecklund Transformations of PDEs"** **Sergey IGONIN**
15.50-16.25	**"Homotopical Geometry and the BV Formalism"** **Frédéric PAUGAM**

16.25-17.00	**"On Deformations of Singular Poisson Algebras"** **Hans-Christian HERBIG**
17.00-17.30	*Coffe Break*
17.30-19.30	**Discussion**
20.00	*Conference Dinner*

Thursday, 16 June 2011

09.05-12.30	**Morning Session: Solution Sigularities of PDEs II**
09.05-09.40	**"The Geometry of Jet Spaces of Pairs of Manifolds"** **Giovanni MORENO**
09.40-10.15	**"Hamilton-Jacobi Diffieties"** **Luca VITAGLIANO**
10.15-10.45	*Coffee Break*
10.45-12.30	**Discussion on Follow-up Activities/Networking/Collaboration**
13.00	*Lunch*
afternoon	*Departure*

5. Final list of participants

Convenors:

1. **Alexandre VINOGRADOV**, University of Salerno, Italy
2. **Giuseppe MARMO**, University of Napoli "Federico II", Italy
3. **Giovanni MORENO**, University of Salerno, Italy
4. **Luca VITAGLIANO**, University of Salerno, Italy

ESF Representative:

5. **Manuel DE LEÓN**, IMAFF CSIC, Spain

Participants:

6. **Aiyalam BALACHANDRAN**, Syracuse University, USA
7. **Evgeny FERAPONTOV**, Loughborough University, UK
8. **Katarzyna GRABOWSKA**, University of Warsaw, Poland
9. **Janusz GRABOWSKI**, Polish Academy of Sciences, Poland
10. **Hans-Christian HERBIG,** Aarhus Universitet, Denmark
11. **Sergey IGONIN**, University of Utrecht, The Netherlands
12. **Vyacheslav KALNITSKY**, St. Petersburg State University, Russia
13. **Jerzy KIJOWSKI**, Polish Academy of Sciences, Poland
14. **Hong Van LE**, Academy of Sciences of Czech Republic, Czech Republic
15. **Michal MARVAN**, Silesian University in Opava, Czech Republic
16. **Frédéric PAUGAM**, Institut de Mathematiques de Jussieu, France
17. **Norbert PONCIN**, University of Luxembourg, Grand-Duchy of Luxembourg
18. **M. Eugenia ROSADO MARÍA**, Universidad Politécnica de Madrid, Spain
19. **Monika Ewa STYPA**, University of Salerno, Italy
20. **Johan VAN DE LEUR**, University of Utrecht, The Netherlands
21. **Silvia VILARIÑO FERNÁNDEZ,** Universidade da Coruña, Spain
22. **Elizaveta VISHNYAKOVA**, Ruhr-Universität Bochum, Germany

6. Statistical information on participants (excuding the ESF Rapporteur)

Age bracket:

20-30	2
30-40	7
40-50	7
50-60	2
60-70	2
over 70	2

Countries of Origin:

Italy	5
Spain	2
USA	1
UK	1
Poland	3
Denmark	1
Russia	1
The Netherlands	2
Czech Republic	2
France	1
Luxembourg	1
Germany	1

Gender Balance:

Men	15
Women	6