



Exploratory Workshop Scheme

Scientific Review Group for Physical and  
Engineering Sciences

ESF Exploratory Workshop on  
**New approaches to Multiple Zeta Values**

Madrid (Spain), September 30 – October 2, 2013

Convened by:  
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**Dominique Manchon <sup>③</sup>**

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**SCIENTIFIC REPORT**

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## 1. Executive summary

The Instituto de Ciencias Matemáticas (ICMAT – Institute of Mathematical Sciences) in Madrid, hosted from September 30 to October 2, 2013, the ESF exploratory workshop “New approaches to Multiple Zeta Values”. The organizing committee included Luis Álvarez-Cónsul (ICMAT-CSIC), José Ignacio Burgos Gil (ICMAT-CSIC), Kurusch Ebrahimi-Fard (ICMAT-CSIC), Marina Logares (ICMAT-UAM), Dominique Manchon (CNRS), and Juanjo Rué Perna (ICMAT-UAM, now FU Berlin, Germany).

This ESF exploratory workshop evolved around the mathematical theory for a particular subset of the real numbers known as multiple zeta values. These numbers come with remarkable families of polynomial identities (with rational coefficients) between them. Multiple zeta values stand at the intersection of several mathematical areas including algebra, combinatorics, algebraic geometry, Lie group theory, and number theory. Moreover, multiple zeta values have strong links with high-energy physics (quantum field theory and string theory).

In order to shed new light on this very active research field in modern mathematics, we invited leading experts working on the theory of multiple zeta values from different perspectives, and with diverse approaches. The main goal was to explore new ideas by giving first class (senior and junior) researchers working in the different fields mentioned above, a common platform through both research talks and open discussion sessions.

The workshop had 20 scientific participants from 6 different EU countries (Austria, France, Germany, Spain, Switzerland, UK). One participant (Prof. K. Yeats, Simon Fraser University, Burnaby) was from Canada. Three researchers, who were visiting ICMAT at the same time (Prof. H. Furusho from Japan, Prof. I. Soudères from Germany, and Prof. U. Kühn from Germany), attended as external participants. Seven participants were young PhD or post-doctoral researchers.

The 3-days workshop consisted of 14 research talks. Each talk was followed by a 15 min. break, which was used for brief discussions. Beside these short breaks, the workshop included four extended 1hr. discussion meetings, which complemented the short breaks. These moderated roundtable sessions played a crucial role during this workshop, since participants –with different backgrounds– could discuss in greater detail the presentations, and also propose research problems, which led to further exchanges of ideas, and eventually new research work. Proposals, open problems and interesting questions were collected on slides, and presented during the discussion sessions.

All talks were very well prepared, especially from the point of view of addressing a mixed audience. Participants engaged quickly into high level discussions of problems presented in talks, as well as roundtable sessions. For more technical discussions, participants could use shared offices, which allowed them to continue the exchange of ideas in greater detail at blackboards, focusing on concrete results in related publications.

The atmosphere of the meeting was exciting and very stimulating. We went together for lunch in a nearby university restaurant. Blocks of talks were followed by coffee breaks. The intense scientific part was complemented in the evening by joint dinners. Though, very often discussions would continue at these occasions. Feedback from the participants was throughout positive, and people seem to have enjoyed a lot this meeting.

As one of the main conclusions, we would like to draw the attention to the upcoming full research trimester on the theory of multiple zeta values and related topics, which is going to be hosted by ICMAT in the second half of 2014. One of the major parts of this trimester is a research-in-teams period of two months, in which we will invite small groups of experts to pursue research projects at ICMAT, accompanied by research talks and introductory lectures. Several of the workshop participants showed great interest, and agreed to join this project. One of the major goals –from a foremost Spanish perspective– is to trigger interest of young post-doctoral researchers to visit ICMAT as well as to attract PhD students. The trimester shall end in December 2014 with the international “Workshop on Multiple Zeta Values, Modular Forms and Elliptic Motives”. The organizing and scientific committees includes L. Schneps, O. Patashnick, H. Gangl, Jürg Kramer, and R. Hain, among others.

## **2. Scientific content of the event**

The theory of multiple zeta values (MZVs) is a recent research topic in mathematics, more precisely, modern number theory. In fact, a systematic study of MZVs only started in the early 1990s with the seminal works of M. Hoffman (U.S. Naval Academy, Annapolis, USA) and D. Zagier (MPIM Bonn, Germany & Collège de France, Paris, France) – although the prehistory can be traced back to L. Euler in the 18th century. We also remark that the first mentioning of the whole set of MZVs appeared under the name “Moule Zeta” in the work of the French mathematician J. Écalle, in his first and second 1981 preprints on resurgent functions.

Since then MZVs and related objects, such as multiple polylogarithms, have turned into a vibrant research field, which touches upon several areas of modern mathematics, including combinatorial algebra (regularized double shuffle relations), algebraic geometry (Grothendieck’s motives), and Lie group theory (Kashiwara-Vergne conjecture). This rich mathematical spectrum is complemented through deep connections with high-energy physics, i.e., perturbative quantum field theory (QFT) as well as string theory.

Amongst the most fundamental properties of MZVs are the relations they satisfy resulting from the fact that they origin from iterated integrals and iterated sums. The holy grail of open problems, roughly speaking, the conjectural algebraic independence of MZVs, over the rationals, modulo a set of well-understood relations is still far from being solved.

In this workshop a rather challenging body of major conjectures on MZVs, as well as several open questions involving aspects in other fields related to MZVs, were examined through a whole range of different approaches.

Let us mention several particularly interesting moments that appeared during the meeting. One of the center aspects of the workshop was the recent striking result due to F. Brown and D. Zagier, in the context of motivic MZVs. Many of the discussions included the mentioning of these works.

The talk by Stieberger (a theoretical physicist from MPI Munich, Germany) started the workshop by presenting new and rather surprising results in the context of open superstring amplitudes. He included several open problems, which were well motivated experimentally, i.e., by computational checks in higher orders on computers. Yeats’ talk on insights into MZVs, from the point of view of graph and matroid theory in the context of QFT complemented Stieberger’s talk from an algebro-combinatorial perspective. Stieberger’s and Yeats’ talks triggered quickly discussions between mathematicians and physicists.

In his talk Manchon presented an open question about "arborified" MZVs, which are suitable linear combinations of MZVs indexed by coloured rooted trees, along the lines of J. Écalle's notion of "arborification". His presentation was received with great interest among the participants. Yeats raised the most interesting reply/question about a possible generalization to finite posets.

In a more recent approach people have tried to understand MZVs from a different point of view, i.e., via so-called multiple Eisenstein-series, which were introduced by Gangl, Kaneko and Zagier in 2006. These are periodic functions on the upper half-plane, which generalize the notion of classical Eisenstein-series, and are well-known to be modular forms. This was the setting of the talk on multiple divisor-functions and  $q$ -analogs of MZVs, given by Bachmann (a young PhD student from Hamburg, Germany). He touched upon several topics that would re-appear in later talks. He particularly contributed a lot to this meeting by asking challenging and interesting questions, which led to many discussions.

Another hot topic, which triggered exciting discussions, was the exploration of algebraic structures underlying MZVs, through the notion of associators, and the related Lie-theoretic Kashiwara-Vergne conjecture. Enriquez's talk explained the basis of multiple elliptic polylogarithms. This is a new theory that may have many consequences. Classical polylogarithms can be seen as degenerations of elliptic polylogarithms. Thus elliptic polylogarithms may shed some light on the properties of classical polylogarithms. Moreover, the multiple divisor sums explained by Bachmann seem to be related to elliptic polylogarithms, although the connection is not completely clear. Finally, there was some discussion on the possible applications of elliptic polylogarithms to string theory in the one loop case. The talks by Alekseev and Schneps were closely related. To summarize, the three talks by Enriquez, Alekseev and Schneps triggered intense interactions, and the three participants engaged into research right on the spot.

Gangl reported on the use of so-called symbol calculus to find functional equations and relations among polylogarithms. This method has found applications in high-energy theoretical physics providing dramatic simplifications in the computation of certain amplitudes related to Feynman diagrams. Patashnick and Agarwala, presented a follow up of this work, where the ideas surrounding the symbol of a polylogarithm are used to construct a complex of cycles based on graphs that might be useful in the study of algebraic cycles. These three talks stirred intense discussion and some controversy.

Sarah Carr explained how to obtain a simplification of the recent proof by Furusho, saying that the Grothendieck-Teichmüller Lie algebra injects into the double shuffle Lie algebra. Her results are based on two combinatorial properties of elements satisfying the stuffle relations. Her talk was closely related to Schneps' talk on the elliptic theory of multiple zeta values.

Bouillot's talk aimed at defining the algebra of so-called multitangent functions, which turns out to be a functional analogue of the algebra of multiple zeta values. He discussed the most important algebraic and analytic properties of these functions, including their consequences for multiple zeta values.

Paycha's work dwelled on the use of the algebraic approach of Connes and Kreimer to perturbative quantum field theory in the study of multi-variable zeta values from summations of lattice points in convex rational cones, called conical zeta values. This allows her to define renormalised conical zeta values at non-positive integers as the Taylor coefficients at zero of

the interpolating holomorphic function. This talk linked well with the presentation given by one of the external participants, H. Furusho, who presented yet another approach to the problem of extending MZVs to non-positive integers. Due to the fact that the results obtained by both approaches differ largely, the discussions were rather intense, with many ideas on how to understand these discrepancies.

### **3. Assessment of the results, contribution to the future direction of the field, outcome**

The workshop was a success in every way. First, the constellation of participants with diverse backgrounds permitted a panoramic view on the whole picture. Second, this grand view led to what this workshop was supposed to be, i.e., an exploratory scientific event. Several intriguing connections surfaced, that triggered instant involvement of the participants into the exchange of ideas, including criticisms, all of which nicely galvanized into high-level scientific interactions.

More precisely, let us mention, for instance, that by joining the talks of Stieberger and Enriquez we identified promising new research objectives. In Stieberger's talk, we learned that MZVs appear in string theory at tree level because, at this level, the string is a projective line minus a finite set of points. The next level of complexity is one loop, where the string is a punctured elliptic curve. Hence, it is likely that elliptic analogues of MZVs, and polylogarithms may appear in string theory at one loop level. In his talk, Enriquez explained his construction of elliptic polylogarithms. Merging both ideas, we plan to study the applications of elliptic polylogarithms in string theory.

In another direction (though related), in the recent work by Bachmann and U. Kühn on multiple divisor functions and multiple Eisenstein-series appears a  $q$ -analogue of multiple zeta values ( $q$ -MZVs). Motivated by Bachmann's talk on this most interesting topic, we started discussions on several other models of  $q$ -MZVs (by Bradley, Ohno-Okuda-Zudilin, Schlesinger, etc.), which induced a most promising ongoing collaboration: J. Castillo-Medina (a young 3<sup>rd</sup> year math student from Valencia), Ebrahimi-Fard and Manchon described the double shuffle picture for the Ohno-Okuda-Zudilin model, and Bachmann proved how to express any modular form in this model. He elaborated and diffused a program (in PARI) for computing  $q$ -MZVs in several of the models mentioned above. An fruitful collaboration is also ongoing between Bachmann, Bouillot and Kühn. Let us also mention the new work of Bouillot on renormalization of multiple zeta values at non-positive arguments, deeply renewing an approach by Manchon and Paycha a couple of years ago.

The organization of a series of follow-up activities has been started. The goal is to profit from the momentum gained through the ESF workshop. One of the convenors, J. I. Burgos, gave a course on MZVs and elliptic curves at the Kosmos Summer School at Humboldt University in Berlin, where experts like, for instance, Schnetz, Brown, Zagier and Kreimer gave also courses, and Bachmann together with young students from Madrid participated.

A MZVs study group just started at the ICMAT. It focuses on motivic aspects of MZVs, and aims at mastering Brown's seminal approach to motivic MZVs, as well as the applications of MZVs and related structures in the context of (perturbative) QFT. In this study group several of the organizers of the ESF meeting participate, including guest researchers and students.

Currently we are deeply immersed at ICMAT in the organization of the 2014 Clay Mathematics Institute Summer School "Periods and Motives: Feynman amplitudes in the 21st century" ([www.icmat.es/summerschool2014/](http://www.icmat.es/summerschool2014/)), with the participation of leading experts such

as Spencer Bloch (Chicago), Hélène Esnault (FU Berlin), Matilde Marcolli (Caltech), Francis Brown (CNRS-IHÉS), Claude Duhr (Durham) and Joseph Ayoub (Zürich).

As we mentioned above, during the workshop the plan to organize a special research trimester on MZVs and related topics solidified. This has been taken further, and this event is going to take place in the fall of 2014 at ICMAT in Madrid. The organizing committee involves, among others, Gangl, Bachmann, Schneps and Patashnick, all of which participated in the ESF workshop at ICMAT. As part of this special research trimester, there will be an international conference on elliptic motives and polylogarithms.

#### 4. Final programme

*Sunday (Sept. 29) Evening*                      *Informal get-together/dinner*

### Monday, 30 September 2013

**09.00-17:00**                      **Session I: MZVs and Physics + Motives I**

09.00-09.10                      **Welcome by Convenor**

**J. I. Burgos** (ICMAT-CSIC, Madrid, Spain)

09.10-09.30                      **Presentation of the European Science Foundation (ESF)**

**Prof. M. Gyllenberg** (Scientific Review Group for Physical and Engineering)

09.30-10.15                      **Presentation 1 "Motivic Multiple Zeta Values, Lie Algebras and Superstring Amplitudes"**

**S. Stieberger** (MPI, Munich, Germany)

10:30-11:00                      *Coffee / Tea Break*

11.00-11.45                      **Presentation 2 "Graph and matroid insights into multiple zeta values in quantum field theory"**

**K. Yeats** (Simon Fraser Univ., Burnaby, Canada)

12.00-12.45                      **Presentation 3 "Arborified multiple zeta values"**

**D. Manchon** (CNRS, Clermont-Ferrand, France)

13.00-13.30                      **Discussion Session I**

13.30-15.00                      *Lunch*

15.00-15.45                      **Presentation 4 "The algebra of multiple divisor-functions and a q-analogue of multiple zeta-values"**

**H. Bachmann** (Univ. Hamburg, Hamburg, Germany)

16.00-17.00                      **Special presentation: "Desingularization of complex multiple zeta-functions and fundamentals of p-adic multiple L-functions"**

**H. Furusho** (Nagoya, Japan)

*Evening*                                      *Informal get-together/dinner*

### Tuesday, 1 October 2013

**09.00-17:00**                      **Session II: MZVs and Lie algebras**

09.00-09.45                      **Presentation 1 "Logarithms and deformation quantization"**

**A. Alekseev** (Univ. Geneva, Geneva, Switzerland)

10:00-10:30                      *Coffee / Tea Break*

10.30-11.15	<b>Presentation 2 "The <math>\gamma</math>-cyclic property in the double shuffle Lie algebra"</b> <b>S. Carr</b> (Ludwig-Maximilians-Univ., Munich, Germany)
11.30-12.15	<b>Presentation 3 "Elliptic associators and MZVs"</b> <b>B. Enriquez</b> (CNRS, Univ. de Strasbourg, Strasbourg, France)
12.30-13.30	<b>Discussion Session II</b>
13.30-15.00	<i>Lunch</i>
15.00-15.45	<b>Presentation 4 "Elliptic theory of multiple zetas"</b> <b>L. Schneps</b> (CNRS, Paris, France)
16.00-17.00	<b>Presentation 5 "The Algebra of Multitangent Functions"</b> <b>O. Bouillot</b> (Paris 11, France)
<i>Evening</i>	<i>Informal get-together/dinner</i>

## Wednesday, 2 October 2013

<b>09.00-17.00</b>	<b>Session III: MZVs and Combinatorics + Motives II</b>
09.00-09.45	<b>Presentation 1 "Multiple polylogarithms in weight 4"</b> <b>H. Gangl</b> (Univ. Durham, Durham, UK)
10:00-10:30	<i>Coffee / Tea Break</i>
10.30-11.15	<b>Presentation 2 "Algebraic cycles, Tate motives, and algebraic graph theory, part I"</b> <b>O. Patashnik</b> (Bristol Univ., Bristol, UK)
11.30-12.15	<b>Presentation 3 "Algebraic cycles, Tate motives, and algebraic graph theory, part II"</b> <b>S. Agarwala</b> (Oxford Univ., Oxford, UK)
12.30-13.30	<b>Discussion Session III</b>
13.30-15.00	<i>Lunch</i>
15.00-15.45	<b>Presentation 4 "Renormalisation of conical zeta values and the Euler-Maclaurin formula"</b> <b>S. Paycha</b> (Univ. Potsdam, Potsdam, Germany)
16.00-17.30	<b>Discussion Session VI: follow-up activities/networking/collaboration</b>
17.30	<i>End of Workshop</i>
<i>Evening</i>	<i>Final informal get-together/dinner</i>

**5. Final list of participants** (name and affiliation is sufficient; the detailed list should be updated on-line directly)

Susama AGARWALA, Oxford Univ., UK

Anton ALEKSEEV, Univ. of Geneva, Switzerland

Luis ÁLVAREZ-CÓNSUL, ICMAT-CSIC, Madrid, Spain

Henrik BACHMANN, Hamburg Univ., Germany

Olivier BOUILLOT, Univ. Paris 11, France

José Ignacio BURGOS GIL, ICMAT-CSIC, Madrid, Spain

Sarah CARR, LMUniv., Munich, Germany

Kurusch EBRAHIMI-FARD, ICMAT-CSIC, Madrid, Spain

Benjamin ENRIQUEZ, CNRS, Strasbourg, France

Herbert GANGL, Durham Univ., UK

Christian KRATTENTHALER, Univ. of Vienna, Austria

Marina LOGARES, ICMAT-CSIC, Madrid, Spain

Dominique MANCHON, CNRS, Clermont-Ferrand, France

Andrea MILLER, Univ. Duisburg-Essen, Germany

Owen PATASHNICK, Bristol Univ., UK

Sylvie PAYCHA, Potsdam Univ., Germany

Juanjo RUÉ PERNA, ICMAT-UAM, Madrid, Spain

Leila SCHNEPS, CNRS, Paris, France

Stephan STIEBERGER, MPI Munich, Germany

Karen YEATS, Simon Fraser Univ., Burnaby, Canada

**6. Statistical information on participants** (age bracket, countries of origin, M/F repartition, etc.) The statistics to be provided under section 6 can also include repartition by scientific specialty if relevant.

20 participants (7 female participants)

6 EU countries:

Austria: Christian KRATTENTHALER (senior)

France: Olivier BOUILLOT (junior), Benjamin ENRIQUEZ (senior), Dominique MANCHON (senior), Leila SCHNEPS (senior)

Germany: Henrik BACHMANN (junior), Sarah CARR (junior), Andrea MILLER (senior), Sylvie PAYCHA (senior), Stephan STIEBERGER (senior)

Spain: José Ignacio BURGOS GIL (senior), Kurusch EBRAHIMI-FARD (senior), Marina LOGARES (junior), Juanjo RUÉ PERNA (junior), ÁLVAREZ-CÓNSUL (senior)

Switzerland: Anton ALEKSEEV (senior)

UK: Susama AGARWALA (junior), Herbert GANGL (senior), Owen PATASHNICK (senior)

1 non-EU country:

Canada: Prof. K. YEATS (senior), Simon Fraser University, Burnaby