Logical formalizations of fuzzy similarity-based reasoning

L. Godo – IIIA-CSIC, Barcelona, Spain

Joint work with
F. Esteva (Barcelona), R. Rodríguez (B. Aires), T. Vetterlein (Linz)

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Outline

- Introduction: uncertainty, fuzziness and truthlikeness
- Graded similarity relations and truthlikeness
- Similarity-based entailment relations
- Logical formalizations
- Conclusions

Uncertainty vs. fuzziness

Possible worlds scenario: W

Ideal situation
(i) complete information about which is the *real world* w_0
(ii) precise concepts: in any world, either $w \models \varphi$ or $w \models \neg\varphi$

$$\mathbf{T} = \{\varphi \mid w_0 \models \varphi\} \quad \mathbf{F} = \{\psi \mid w_0 \models \neg\psi\}$$

Some more realistic situations:

Uncertainty about w_0: incomplete information but still precise concepts

- the real world is in $K \subset W$

$$\mathbf{T} = \{\varphi \mid \forall w \in K, w \models \varphi\} \quad \mathbf{F} = \{\psi \mid \forall w \in K, w \models \neg\psi\}$$
$$\mathbf{U} = \{\varphi \mid \varphi \notin \mathbf{T}, \varphi \notin \mathbf{F}\}$$

- w_0 as a random variable with a probability function $P : 2^W \to [0, 1]$

$$0 \leq Prob(\varphi) = P(\{w \mid w \models \varphi\}) \leq 1$$

Uncertainty vs. fuzziness

Fuzziness:
(i) complete information: the real world is w_0
(ii) gradual concepts: in any world, $w(\varphi) \in [0, 1]$

many-valued worlds, intermediate degrees of truth:

$$0 \leq truth(\varphi) = w_0(\varphi) \leq 1$$

Mathematical fuzzy logics:
- $[0, 1]$: usual choice of truth-value set (standard semantics)
- truth-functionality assumption
- logics of comparative truth: $w(\varphi \to \psi) = 1$ iff $w(\varphi) \leq w(\psi)$

More complex scenarios: incomplete information + imprecise concepts

- E.g. the real world is in $K \subseteq W$:

  - $\min\{w(\varphi) \mid w \in K\} \leq truth(\varphi) \leq \max\{w(\varphi) \mid w \in K\}$

- etc.

Truthlikeness

Truthlikeness $\neq$ Uncertainty, Fuzziness

φ_1: there are 300 steps to the top of *La Torre di Pisa*

φ_2: there are 1000 steps to the top of *La Torre di Pisa*

In the real world w_0 both are false (there are 296!),

... but clearly φ_1 provides a more accurate description of w_0 than φ_2.

Indeed, 300 is more similar to 296 than 1000.

"φ_1 is closer to be true (more truth-like) than φ_2"

Truthlikeness

- (G. Oddie, Stanford Encyclopedia of Philosophy)

  Truthlikeness: *"... classify propositions according to their closeness to the truth, their degree of truthlikeness or verisimilitude ... give an adequate account of the concept and to explore its logical properties and its applications . . . to epistemology and methodology"*

  - Popper, Tichý, Hilpinen, Niiniluoto, ...

- A further (independent) dimension to be additionally considered to models dealing with imperfect information (uncertainty, fuzziness, nonmonotonicity, ect.)

- Our approach: a fuzzy similarity-based account of truthlikeness

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A (graded) similarity-based account of truthlikeness

Equip the set of possible worlds W with some kind of metric or, dually, similarity measure

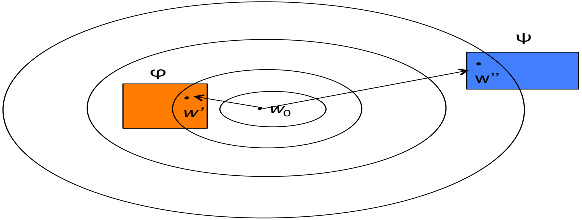
Here, a $\otimes$-similarity relation on W is a mapping $S : W \times W \to [0,1]$
$S(w, w') :=$ how much similar is w to w'

- **Reflexivity**: $S(u, u) = 1$
  Separation: $S(u, v) = 1$ only if $u = v$
- **Symmetry**: $S(u, v) = S(v, u)$
- **$\otimes$-Transitivity**: $S(u, v) \otimes S(v, w) \leq S(u, w)$

- when $x \otimes y = \max(x + y - 1, 0)$, then $\delta = 1 - S$ is a distance

Weaker notions: closeness relations (Refl),
proximity, tolerance relations (Refl + Sim)

- a more informed scenario: complete information w_0 + precise concepts + a similarity S between possible worlds



Both φ and ψ are false at w_0 but

φ is closer to be true (more truthlike) than ψ

and now this can be quantified:

$$\textit{truthlikeness}(\varphi) = \max\{S(w_0, w') \mid w' \models \varphi\} \geq \max\{S(w_0, w'') \mid w'' \models \psi\} = \textit{truthlikeness}(\psi)$$

A more fine-grained representation and reasoning framework:

- In the enriched ideal scenario (w_0 + precise concepts + similarity) we still have the partition:

$$\mathbf{T} = \{\varphi \mid w_0 \models \varphi\} \quad \mathbf{F} = \{\psi \mid w_0 \models \neg\psi\}$$

but now we can refine it: $\mathbf{F} = \cup_{\alpha<1}$ α-Truthlike, where:

$$\alpha\text{-Truthlike} = \{\psi \mid \mathit{truthlikeness}(\psi) = \alpha\}$$

- More generally, given a theory (epistemic state), one may identify which consequences are closer (more truth-like) to hold than others

Aim of our work: logical formalizations of some patterns of (degree-based) similarity-based reasoning

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- Introduction: uncertainty, fuzziness and truthlikeness
- Graded similarity relations and truthlikeness
- Similarity-based entailments
  - approximate entailment
  - strong entailment
- Logical formalizations
- Conclusions

Focus: two kinds of entailment tolerant to small changes

Given $\varphi \models \psi$,

1) How to define $\varphi \models^* \psi'$ such that:

If ψ' is similar to ψ, $\varphi \models \psi'$ remains still "valid"

- the more ψ' is similar to ψ, the more truthlike is ψ' when φ is true

2) How to define $\varphi \approx^* \psi$ such that:

If φ' is similar to φ, $\varphi' \models \psi$ remains still "valid"

- the less φ' is similar to φ, the stronger $\approx^*$ should be

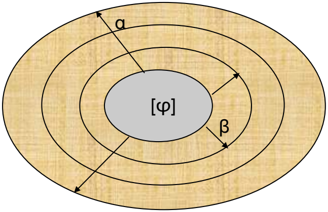
Approximate entailment

$S : W \times W \to [0,1]$

$\Rightarrow$ spheres around the set of models of a proposition $[\varphi]$

$$U_\alpha([\varphi]) = \{w \in W \mid \text{ exists } w' \in [\varphi] \text{ and } S(w', w) \geq \alpha\}$$

$$[\varphi] = U_1([\varphi]) \subseteq \ldots \subseteq U_\alpha([\varphi]) \subseteq \ldots \subseteq U_0([\varphi]) = W$$



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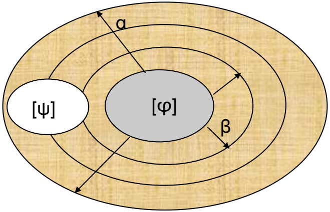


$\psi \not\models \varphi$, but $[\psi] \subseteq U_\alpha([\varphi])$

ψ α-approximately entails φ

$\not\models \varphi \wedge \psi$, but $[\psi] \cap U_\beta([\varphi]) \neq \emptyset$

ψ and φ are β-consitent

$I_S(\varphi \mid \psi) = \sup\{\alpha \mid [\psi] \subseteq U_\delta([\varphi])\}$
$C_S(\varphi \mid \psi) = \sup\{\delta \mid [\psi] \cap U_\delta([\varphi]) \neq \emptyset\}$

Approximate entailment: characterization

Approximate entailment (cf. DEGGP,97): Given a $\otimes$-similarity $S : W \times W \to V$, with $V \subseteq [0,1]$, define:

$$\varphi \models^{\alpha}_{S} \psi \quad \text{iff} \quad [\varphi] \subseteq U_{\alpha}([\psi])$$
$$\text{iff} \quad \text{for all } \omega,\ \omega \models \varphi \text{ implies } \exists \omega' : \omega \models \psi \text{ and } S(\omega, \omega') \geq \alpha$$

Characterizing properties:

(1)	**Supraclassicality:**	if $\varphi \models \psi$ then $\varphi \models^{\alpha} \varphi$ (in particular $\varphi \models^{1} \varphi$)
(2)	**Nestedness:**	if $\varphi \models^{\alpha} \psi$ and $\beta \leq \alpha$ then $\varphi \models^{\beta} \psi$;
(3)	**Left OR:**	$\varphi \vee \chi \models^{\alpha} \psi$ iff $\varphi \models^{\alpha} \psi$ and $\chi \models^{\alpha} \psi$;
. . .		
(6)	**Symmetry:**	if $\varphi \models^{\alpha} \psi$ then $\psi \models^{\beta} \varphi$, if $U_{\alpha}([\varphi]), U_{\alpha}([\psi])$ singletons
(7)	$\otimes$-**Transitivity:**	if $\varphi \models^{\alpha} \chi$ and $\chi \models^{\beta} \psi$ then $\varphi \models^{\alpha \otimes \beta} \psi$;

$$\boxed{\varphi \models \psi \quad \text{implies} \quad \varphi \models^{\alpha}_{S} \psi}$$

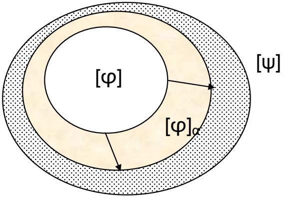
From Approximate to Strong entailment

Approximate reasoning: derivation of approximate consequences

If φ then approximately ψ

Strong reasoning: inferences tolerant to small changes in the premise

If approximately φ then ψ



$\varphi \mathrel{|\!\approx}^{\alpha}_{S} \psi$ iff $U_\alpha([\varphi]) \subseteq [\psi]$

stronger than classical $\models$

$J_S(\psi \mid \varphi) = \sup\{\alpha \mid U_\alpha([\varphi]) \subseteq [\psi]\}$

Strong entailment: characterization

Definition: Given a $\otimes$-similarity relation $S : W \times W \to V$

$$\varphi \approx\!\!\!|_S^{\alpha} \psi \quad \text{iff} \quad U_\alpha([\varphi]) \subseteq [\psi]$$
$$\text{iff} \quad \text{for all } \omega,\ \omega \models_S^{\alpha} \varphi \text{ implies } \omega \models \psi$$

Characterizing properties:

(1)	**Nestedness:**	if $\varphi \approx\!\!\!|_S^{\alpha} \psi$ and $\beta \geq \alpha$ then $\varphi \approx\!\!\!|_S^{\beta} \psi$;
(2)	**Lower bound:**	$\varphi \approx\!\!\!|_S^{0} \psi$ iff either $\models \neg\varphi$ or $\models \psi$
(3)	**Upper bound:**	$\varphi \approx\!\!\!|_S^{1} \psi$ iff $\varphi \models \psi$
(4)	**min-Transitivity:**	if $\varphi \approx\!\!\!|_S^{\alpha} \psi$ and $\psi \approx\!\!\!|_S^{\beta} \chi$ then $\varphi \approx\!\!\!|_S^{\min(\alpha,\beta)} \chi$;
(5)	**Left OR:**	$\varphi \vee \chi \approx\!\!\!|_S^{\alpha} \psi$ iff $\varphi \approx\!\!\!|_S^{\alpha} \psi$ and $\chi \approx\!\!\!|_S^{\alpha} \psi$;
(6)	**Right AND:**	$\chi \approx\!\!\!|_S^{\alpha} \varphi \wedge \psi$ iff $\chi \approx\!\!\!|_S^{\alpha} \varphi$ and $\chi \approx\!\!\!|_S^{\alpha} \psi$.
(7)	**Contraposition:**	if $\varphi \approx\!\!\!|_S^{\alpha} \psi$ then $\neg\psi \approx\!\!\!|_S^{\alpha} \neg\varphi$

...

$$\boxed{\varphi \approx\!\!\!|_S^{\alpha} \psi \quad \text{implies} \quad \varphi \models \psi \quad \text{implies} \quad \varphi \models_S^{\alpha} \psi}$$

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Logics of approximate and strong entailments

Aim: encode graded entailments "$\varphi \models^{\alpha}_{S} \psi$" and "$\varphi \approx^{\alpha}_{S} \psi$" as syntactic objects by conditional-like formulas

Language(s):
- if φ, ψ are CPC propositions and $\alpha \in C \subset [0,1]$, then

$$\varphi >_{\alpha} \psi \qquad \varphi \succ_{\alpha} \psi$$

are LAE and LSE graded conditional formulas resp.

- LAE language: built from conditionals $\varphi >_{\alpha} \psi$ and CPC connectives;
- LSE language: built from conditionals $\varphi \succ_{\alpha} \psi$ and CPC connectives;

(no nested conditional formulas !!)

- LASE language: analogously built with both kinds of conditionals

Semantics: Kripke-like models $M = (W, e, S)$, where:

- W set of possible worlds
- e : *Propositions* $\to 2^W$
- $S : W \times W \to V \subset [0, 1]$ is a $\otimes$-similarity

$$M \models \varphi >_\alpha \psi \quad \text{if} \quad e(\varphi) \subseteq U_\alpha(e(\psi))$$

$$M \models \varphi \succ_\alpha \psi \quad \text{if} \quad U_\alpha(e(\varphi)) \subseteq [\psi]$$

$M \models \Phi$ is otherwise defined like in CPC

- CPC formulas φ can be interpreted into LAE (resp. LSE) as $\top >_1 \varphi$ (resp. $\top \succ_1 \varphi$)

LAE fragment: a logic of approximate entailment

Axioms and Rule:

(A1) $\phi >_1 \psi$, if $\phi \to \psi$ is a tautology of CPL

(A2) $(\phi >_\alpha \psi) \to (\phi >_\beta \psi)$, where $\alpha \geq \beta$

(A3) $(\phi >_0 \psi) \vee (\psi >_1 \bot)$

(A4) $(\phi >_\alpha \bot) \to (\phi >_1 \bot)$

(A5) $(\delta >_\alpha \epsilon) \to (\epsilon >_\alpha \delta) \vee (\delta >_1 \bot)$, where δ, ϵ are m.e.c.'s

(A6) $(\phi >_\alpha \chi) \wedge (\psi >_\alpha \chi) \to (\phi \vee \psi >_\alpha \chi)$

(A7) $(\epsilon >_\alpha \phi \vee \psi) \to (\epsilon >_\alpha \phi) \vee (\epsilon >_\alpha \psi)$, where ϵ is a m.e.c.

(A8) $(\phi >_1 \psi) \to (\phi \wedge \neg\psi >_1 \bot)$

(A9) $(\phi >_\alpha \psi) \wedge (\psi >_\beta \chi) \to (\phi >_{\alpha \odot \beta} \chi)$

(A10) LAE-formulas obtained by uniform replacements of variables in CPL-tautologies by LAE graded conditionals

(MP) Modus Ponens

Completeness: $T \vdash_{LAE} \Phi$ iff $T \models_{LAE} \Phi$

LSE fragment: a logic of strong entailment

Axioms and Rule:

(S1) $\phi \succ_1 \psi$, if $\phi \to \psi$ is a tautology of CPL

(S2) $\bot \succ_0 \phi$, $\quad \phi \succ_0 \top$

(S4) $(\phi \succ_0 \psi) \to (\phi \succ_1 \bot) \vee (\top \succ_1 \psi)$

(S5) $(\phi \succ_\alpha \psi) \to (\phi \succ_\beta \psi)$, where $\alpha \leq \beta$

(S6) $(\phi \succ_\alpha \psi) \wedge (\phi \succ_\alpha \chi) \to (\phi \succ_\alpha \psi \wedge \chi)$

(S7) $(\phi \succ_\alpha \chi) \wedge (\psi \succ_\alpha \chi) \to (\phi \vee \psi \succ_\alpha \chi)$

(S8) $(\phi \succ_\alpha \psi) \to (\neg\psi \succ_\alpha \neg\phi)$

(S9) $(\phi \succ_\alpha \psi) \wedge (\psi \succ_\beta \chi) \to (\phi \succ_{\min\{\alpha,\beta\}} \chi)$

(S10) $(\phi \succ_{\alpha \odot \beta} \psi) \to (\epsilon \succ_\alpha \neg\phi) \vee (\epsilon \succ_\beta \psi)$, where ϵ is a m.e.c.

(A10) LSE-formulas obtained by uniform replacements of variables in CPL-tautologies by LSE graded conditionals

(MP) Modus Ponens

Completeness: $\mathcal{T} \vdash_{LSE} \Phi$ iff $\mathcal{T} \models_{LSE} \Phi$

LASE: merging LAE and LSE

Axioms and Rule:

(AS0) Axioms of LAE and LSE

(AS1) $(\phi >_1 \psi) \leftrightarrow (\phi \succ_1 \psi)$

(AS2) $(\phi >_\alpha \psi) \wedge (\psi \succ_\alpha \chi) \rightarrow (\phi >_1 \chi)$

(AS3) $(\epsilon >_\alpha \delta) \leftrightarrow \neg(\delta \succ_\alpha \neg\epsilon)$, where ϵ, δ are m.e.c.'s

(AS4) Given a tautology of CPL, the statement resulting from a uniform replacement of the atoms by graded LAE-implications or graded LSE-implications is an axiom.

(MP) Modus Ponens

Completeness: $\mathcal{T} \vdash_{LASE} \Phi$ iff $\mathcal{T} \models_{LASE} \Phi$

Conclusions

- From a KR perspective, a graded similarity-based account of truthlikeness enriches representation, reasoning and even decision capabilities
- A further (independent) dimension to be additionally considered to models dealing with imperfect information (uncertainty, fuzziness, nonmonotonicity, ect.)
- Well-known links to a restricted form of vagueness / fuzziness:
  a set of prototypes A + a similarity S = a vague/fuzzy concept A^*:
  $\mu^*_A(u) = \sup\{S(u, v) \mid v \in A\}$
- Clear relation to fuzzy modal logics:
  a similarity relation on worlds as a (graded) accessibility relation $\longrightarrow$ fuzzy modal operators $\Diamond\varphi$
  - approximate conditional: $\varphi \to \Diamond\psi$
  - strong conditional: $\Diamond\varphi \to \psi$
- Related formalisms: morpho-logics (BL), belief change, preference representation (LL), logics of metric spaces (K...)

Thank you !