Proof systems for dependence and independence logic

Pietro Galliani

Institute for Logic, Language and Computation
Universiteit van Amsterdam

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Outline

1. Logics of imperfect information
2. General models
3. The proof system

Hodges' Semantics for First Order Logic

Teams

A team X is a set of assignments over the same first order model and over a finite set $\mathrm{Dom}(X)$ of variables.

- If α literal, $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$;
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$;
- $M \models_X \phi \vee \psi$ iff $X = Y \cup Z$, $M \models_Y \phi$ and $M \models_Z \psi$;
- $M \models_X \exists x \phi$ iff $\exists H : X \to \mathcal{P}(\mathrm{Dom}(M))$ s.t. $M \models_{X[H/x]} \phi$;
- $M \models_X \forall x \phi$ iff $M \models_{X[M/x]} \phi$.

Aside: Hodges Semantics and Game Theoretic Semantics

Teams correspond to *sets of possible states* in the subgames of the semantic game.

Hodges' Semantics for First Order Logic

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Dependence Logic (Väänänen)

Dependence Atoms

$M \models_X =\!(\vec{t}_1, t_2)$ if and only if for all $s, s' \in X$, if s and s' coincide over $\vec{t}_1$ then they coincide over t_2 too (t_2 is a function of $\vec{t}_1$ in X).

Dependence Logic

Dependence Logic = First Order Logic + Dependence Atoms.

"Equivalent" to IF Logic and Branching Quantifier Logic

There exists translations between these logics (wrt sentences).

Independence Logic (Grädel and Väänänen)

Independence Atoms (Grädel, Väänänen)

$M \models_X \vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$ if and only if, for all $s, s' \in X$ such that $\vec{t}_1\langle s \rangle = \vec{t}_1\langle s' \rangle$ there exists a $s'' \in X$ such that

$$\vec{t}_1\langle s''\rangle\vec{t}_2\langle s''\rangle = \vec{t}_1\langle s\rangle\vec{t}_2\langle s\rangle,\ \vec{t}_1\langle s''\rangle\vec{t}_3\langle s''\rangle = \vec{t}_1\langle s'\rangle\vec{t}_3\langle s'\rangle.$$

Independence Logic (Grädel, Väänänen)

Independence Logic = First Order Logic + Independence Atoms.

Inclusion/Exclusion Logic (Galliani)

Inclusion Atoms

$M \models_X \vec{t}_1 \subseteq \vec{t}_2$ if and only if for all $s \in X$ there exists a $s' \in X$ such that

$$\vec{t}_1\langle s\rangle = \vec{t}_2\langle s'\rangle;$$

Exclusion Atoms

$M \models_X \vec{t}_1 \mid \vec{t}_2$ if and only if, for all $s, s' \in X$,

$$\vec{t}_1\langle s\rangle \neq \vec{t}_2\langle s'\rangle.$$

Proof theory for logics of imperfect information?

All of these logics are undecidable!

They are all equivalent to Σ_1^1 over sentences...

We can consider fragments, however...

Väänänen: Proof system for *first order* consequences of *dependence logic* formulas.

... or perhaps we can weaken the semantics?

Example: Second order logic with Henkin semantics.

Idea suggested by Väänänen.

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General premodels

General premodels

A *general premodel* of signature Σ is a pair $(M, \mathcal{G})$, where M is a first order model and $\mathcal{G}$ is a set of teams (relations).

Semantics over premodels

$\mathbf{P} = (M, \mathcal{G})$ general premodel, and let $X \in \mathcal{G}$. Then

- Usual rules for atoms and literals;
- $\mathbf{P} \models_X \phi \wedge \psi$ iff $\mathbf{P} \models_X \phi$ and $\mathbf{P} \models_X \psi$;
- $\mathbf{P} \models_X \phi \vee \psi$ iff $X = Y \cup Z$, $Y, Z \in \mathcal{G}$, $\mathbf{P} \models_Y \phi$ and $\mathbf{P} \models_Z \psi$;
- $\mathbf{P} \models_X \exists x \phi$ iff $\exists H$ s.t. $X[H/x] \in \mathcal{G}$ and $\mathbf{P} \models_{X[H/x]} \phi$;
- $\mathbf{P} \models_X \forall x \phi$ iff $X[M/x] \in \mathcal{G}$ and $\mathbf{P} \models_{X[M/x]} \phi$.

General models

General models

A general premodel $(M, \mathcal{G})$ is a *general model* if and only if $\mathcal{G}$ contains all teams corresponding to relations definable in first order logic (with parameters) over $(M, \vec{\text{Rel}}(\mathcal{G}))$.

Least general models

$(M, \mathcal{L})$ is a *least general model* if and only if

$$\mathcal{L} = \{\|\theta(\vec{x}, \vec{m})\|_M : \theta \in \textbf{FOL}, \vec{m} \in \text{Dom}(M)\}.$$

From least general model to general models

An easy (but useful) result

Let $\mathbf{P} = (M, \mathcal{G})$, $\mathbf{P}' = (M, \mathcal{G}')$, and $\mathcal{G} \subseteq \mathcal{G}'$. Then

$$X \in \mathcal{G}, \mathbf{P} \models_X \phi \Rightarrow \mathbf{P}' \models_X \phi$$

A consequence

$(M, \mathcal{G})$ general model, $(M, \mathcal{L})$ least general model. Then

$$X \in \mathcal{L}, (M, \mathcal{L}) \models_X \phi \Rightarrow (M, \mathcal{G}) \models_X \phi.$$

Validity

Validity

Let ϕ be a independence logic formula. Then

- FUL $\models \phi$ if and only if $M \models_X \phi$ for all first-order models M, according the usual semantics.
- GEN $\models \phi$ if and only if $\mathbf{G} \models_X \phi$ for all general models $\mathbf{G} = (M, \mathcal{G})$ and all $X \in \mathcal{G}$.
- LEA $\models \phi$ if and only if $\mathbf{L} \models_X \phi$ for all least general models $\mathbf{G} = (M, \mathcal{G})$ and all $X \in \mathcal{G}$.

A theorem

For all independence logic formulas ϕ,

$$(\text{LEA} \models \phi \Leftrightarrow \text{GEN} \models \phi) \Rightarrow \text{FUL} \models \phi.$$

3-sequents

3-sequents

A *3-sequent* is an expression of the form $\Gamma(\vec{p}) \mid \theta(\vec{x}, \vec{p}) \vdash \phi(\vec{x})$, where

- $\Gamma(\vec{p})$ is a finite set of first order formulas;
- $\theta(\vec{x}, \vec{p})$ is a first order formula;
- $\phi(\vec{x})$ is an independence logic formula.

Valid 3-sequents

$\Gamma(\vec{p}) \mid \theta(\vec{x}, \vec{p}) \vdash \phi(\vec{x})$ is *valid* if and only if, for all least general model $\mathbf{L} = (M, \mathcal{L})$ and for all $\vec{m}$,

$$M \models \Gamma(\vec{m}) \Rightarrow \mathbf{L} \models_{\|\theta(\vec{x}, \vec{m})\|_M} \phi.$$

The proof system: Axioms

Axioms for literals

PS-FO: For all first order literals $\alpha(\vec{x})$ and all first order $\theta(\vec{p}, \vec{x})$,

$$\forall \vec{x}(\theta(\vec{p}, \vec{x}) \rightarrow \alpha(\vec{x})) \mid \theta(\vec{p}, \vec{x}) \vdash \alpha(\vec{x});$$

PS-dep: For all $\vec{t}(\vec{x})$, $t'(\vec{x})$ and for all first order $\theta(\vec{p}, \vec{x})$,

$$\forall \vec{x}_1 \vec{x}_2 (\theta(\vec{p}, \vec{x}_1) \wedge \theta(\vec{p}, \vec{x}_2) \wedge \vec{t}(\vec{x}_1) = \vec{t}(\vec{x}_2) \rightarrow \\ \rightarrow t'(\vec{x}_1) = t'(\vec{x}_2)) \mid \theta(\vec{p}, \vec{x}) \vdash =\!(\vec{t}, t');$$

Note: A similar rule was found earlier by Väänänen.

The proof system: Axioms

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$$\forall \vec{x}(\theta(\vec{p}, \vec{x}) \rightarrow \alpha(\vec{x})) \mid \theta(\vec{p}, \vec{x}) \vdash \alpha(\vec{x});$$

PS-indep: For all $\vec{t}_1(\vec{x})$, $\vec{t}_2(\vec{x})$ and $\vec{t}_3(\vec{x})$ and for all first order $\theta(\vec{p}, \vec{x})$,

$$\forall \vec{x}_1 \vec{x}_2((\theta(\vec{p}, \vec{x}_1) \wedge \theta(\vec{p}, \vec{x}_2) \wedge \vec{t}_1(\vec{x}_1) = \vec{t}_1(\vec{x}_2)) \rightarrow$$
$$\rightarrow \exists \vec{x}_3(\theta(\vec{p}, \vec{x}_3) \wedge \vec{t}_1\vec{t}_2(\vec{x}_3) = \vec{t}_1\vec{t}_2(\vec{x}_1) \wedge$$
$$\wedge \vec{t}_1\vec{t}_3(\vec{x}_3) = \vec{t}_1\vec{t}_3(\vec{x}_2))) \mid \theta(\vec{p}, \vec{x}_3) \vdash \vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3;$$

The proof system: Axioms

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PS-inc: For all $\theta(\vec{p}, \vec{x})$, $\vec{t}_1(\vec{x})$ and $\vec{t}_2(\vec{x})$

$$\forall \vec{x}_1(\theta(\vec{p}, \vec{x}_1) \rightarrow \exists \vec{x}_2(\theta(\vec{p}, \vec{x}_2) \wedge \vec{t}_1(\vec{x}_1) = \vec{t}_2(\vec{x}_2))) \mid$$
$$\mid \theta(\vec{p}, \vec{x}) \vdash \vec{t}_1 \subseteq \vec{t}_2;$$

The proof system: Axioms

Axioms for literals

PS-FO: For all first order literals $\alpha(\vec{x})$ and all first order $\theta(\vec{p}, \vec{x})$,

$$\forall \vec{x}(\theta(\vec{p}, \vec{x}) \rightarrow \alpha(\vec{x})) \mid \theta(\vec{p}, \vec{x}) \vdash \alpha(\vec{x});$$

PS-exc: For all $\theta(\vec{p}, \vec{x})$, $\vec{t}_1(\vec{x})$ and $\vec{t}_2(\vec{x})$

$$\forall \vec{x}_1 \vec{x}_2((\theta(\vec{p}, \vec{x}_1) \wedge \theta(\vec{p}, \vec{x}_2)) \rightarrow \vec{t}_1(\vec{x}_1) \neq \vec{t}_2(\vec{x}_2)) \mid$$
$$\mid \theta(\vec{p}, \vec{x}) \vdash \vec{t}_1 \mid \vec{t}_2;$$

The proof system: Rules for connectives

Rules for connectives

PS-$\vee$: If $\Gamma_1 \mid \theta_1 \vdash \phi_1$ and $\Gamma_2 \mid \theta_2 \vdash \phi_2$ then, for all θ,

$$\Gamma_1, \Gamma_2, \forall \vec{x}(\theta \leftrightarrow (\theta_1 \vee \theta_2)) \mid \theta \vdash \phi_1 \vee \phi_2;$$

PS-$\wedge$: If $\Gamma_1 \mid \theta \vdash \phi_1$ and $\Gamma_2 \mid \theta \vdash \phi_2$ then $\Gamma_1, \Gamma_2 \mid \theta \vdash \phi_1 \wedge \phi_2$;

PS-$\exists$: If $\Gamma \mid \theta' \vdash \phi$ then, for all θ,

$$\Gamma, \forall \vec{x}(\exists y \theta' \leftrightarrow \exists y \theta) \mid \theta \vdash \exists y \phi;$$

PS-$\forall$: If $\Gamma \mid \theta' \vdash \phi$ then, for all θ,

$$\Gamma, \forall \vec{x}(\theta' \leftrightarrow \exists y \theta) \mid \theta \vdash \forall y \phi.$$

The proof system: Additional rules

Additional rules

PS-ent: If $\Gamma \mid \theta \vdash \phi$ and $\bigwedge \Gamma' \models \bigwedge \Gamma$ holds in First Order Logic then $\Gamma' \mid \theta \vdash \phi$;

PS-depar: If $\Gamma \mid \theta \vdash \phi$ and p is a parameter variable which does not occur free in θ then $\exists p \bigwedge \Gamma \mid \theta \vdash \phi$;

PS-split: If $\Gamma_1 \mid \theta \vdash \phi$ and $\Gamma_2 \mid \theta \vdash \phi$ then $(\bigwedge \Gamma_1) \vee (\bigwedge \Gamma_2) \mid \theta \vdash \phi$.

The main result

The above axiom system is sound and complete for valid 3-sequents.

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The main result

The above axiom system is sound and complete for valid 3-sequents.

The proof system: Additional rules

A further result

Let $\phi(\vec{x})$ be an independence logic formula, and let R be a relation symbol not in it. Then

$$\text{LEA} \models \phi(\vec{x}) \Leftrightarrow \emptyset \mid R(\vec{x}) \vdash \phi(\vec{x}) \text{ is valid.}$$

So, in conclusion...

We now have a proof system for independence logic (with respect to a weaker semantics).

The proof system: Additional rules

A further result

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The end

The end (for now...)

Extra slides: Completeness

A lemma

Suppose that $\mathbf{L} = (M, \mathcal{L}) \models_{||\theta(\vec{x},\vec{m})||_M} \phi(\vec{x})$. Then there exists a finite $\Gamma(\vec{p})$ such that $\Gamma(\vec{p}) \mid \theta(\vec{x}, \vec{p}) \vdash \phi$ is provable and $M \models \Gamma(\vec{m})$.

Proof.

By induction. □

Extra slides: Completeness

Completeness

If $\Gamma(\vec{p}) \mid \theta(\vec{x}, \vec{p}) \vdash \phi(\vec{x})$ is valid, it is provable in our system.

Proof (1).

By the lemma, for any suitable first order model M and for every $\vec{m}$ with $M \models \Gamma(\vec{m})$ there exists a $\Gamma_{M,\vec{m}}(\vec{p})$ s.t.

1. $\Gamma_{M,\vec{m}}(\vec{p}) \mid \theta(\vec{x}, \vec{p}) \vdash \phi(\vec{x})$ is provable;
2. $M \models \Gamma_{M,\vec{m}}(\vec{m})$.

Now consider the first order theory

$$T(\vec{p}) = \left\{\bigwedge \Gamma(\vec{p})\right\} \cup \left\{\neg \bigwedge \Gamma_{M,\vec{m}}(\vec{p}) : M \text{ countable}, M \models \Gamma(\vec{m})\right\}$$

□

Extra slides: Completeness

Proof (2).

$$T(\vec{p}) = \left\{\bigwedge \Gamma(\vec{p})\right\} \cup \left\{\neg \bigwedge \Gamma_{M,\vec{m}}(\vec{p}) : M \text{ countable}, M \models \Gamma(\vec{m})\right\}$$

$T(\vec{p})$ is unsatisfiable: if $M \models T(\vec{m})$ then $\exists\, M_0 \subseteq M$ countable s.t. $M_0 \models T(\vec{m})$. Then $(M_0, \mathcal{L}) \models_{\|\theta(\vec{x},\vec{m})\|_{M_0}} \phi(\vec{x})$, and hence $M_0 \models \Gamma_{M_0,\vec{m}}$.

Therefore, $\bigwedge \Gamma(\vec{p})$ implies $\bigvee_{i=1}^{n} \left(\bigwedge \Gamma_{M_i,\vec{m}_i}(\vec{p})\right)$; but on the other hand, by the split rule, $\bigvee_{i=1}^{n} \left(\bigwedge \Gamma_{M_i,\vec{m}_i}(\vec{p})\right) \mid \theta(\vec{x},\vec{p}) \vdash \phi(\vec{x})$ is provable. □

Extra slides: Completeness

Proof (3).

- $\Gamma(\vec{p})$ implies $\bigvee_{i=1}^{n}\left(\bigwedge \Gamma_{M_i,\vec{m}_i}(\vec{p})\right)$;
- $\bigvee_{i=1}^{n}\left(\bigwedge \Gamma_{M_i,\vec{m}_i}(\vec{p})\right) \mid \theta(\vec{x},\vec{p}) \vdash \phi(\vec{x})$ is provable.

But then, by the entailment rule, $\Gamma(\vec{p}) \mid \theta(\vec{x},\vec{p}) \vdash \phi(\vec{x})$ is provable. □