Implications in team semantics setting

Fan Yang

University of Helsinki, Finland

LogICCC Final Conference
Berlin, September 15-18, 2011

Outline

Dependence Logic

$\mathbf{D} = \mathbf{FO} + {=}(x_1, \ldots, x_n, y)$

Well-formed formulas of **D** (in negation normal form) are given by the following grammar

$$\phi ::= \alpha \mid {=}(x_1, \ldots, x_n, y) \mid \neg{=}(x_1, \ldots, x_n, y) \mid \phi \wedge \phi \mid \phi \otimes \phi \mid \forall x \phi \mid \exists x \phi$$

where α is a first order literal.

Team Semantics

Let X be a team and M an L-structure.

- $M \models_X \alpha$ with α first-order literal iff $M \models_s \alpha$ for all $s \in X$
- $M \models_X =\!(x_1, \cdots, x_n)$ iff for all $s, s' \in X$ such that $s(x_1) = s'(x_1), \cdots, s(x_{n-1}) = s'(x_{n-1})$, we have $s(x_n) = s'(x_n)$
- $M \models_X \neg =\!(x_1, \cdots, x_n)$ iff $X = \emptyset$
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$
- $M \models_X \phi \otimes \psi$ iff $X = Y \cup Z$ s.t. $M \models_Y \phi$ and $M \models_Z \psi$
- $M \models_X \exists x \phi$ iff $M \models_{X(F/x)} \phi$ for some $F : X \to M$
- $M \models_X \forall x \phi$ iff $M \models_{X(M/x)} \phi$

Constancy dependence atom

$$M \models_X =\!(x) \text{ iff for all } s, s' \in X\ s(x) = s'(x).$$

	$\cdots$	x	$\cdots$
s_0		a	
s_1		a	
s_2		a	
s_3		a	
s_4		a	
s_5		a	

Important Properties of **D**

Theorem (Downwards Closure)

For any formula ϕ of **D**, *if $M \models_X \phi$ and $Y \subseteq X$, then $M \models_Y \phi$.*

Theorem (Empty Team Property)

Empty team satisfies every formula $\phi \in$ **D** *in every model M, i.e. $M \models_\emptyset \phi$ for every $\phi \in$* **D** *and every M.*

Expressive power of **D**

On sentence level
[Enderton, Walkoe, Väänänen]

Σ^1_1 **D**

Expressive power of **D**

On formula level

Theorem (Kontinen, Väänänen)

Restricted to nonempty teams, open formulas of **D** *are equivalent to* Σ^1_1 *downwards monotone sentences with a new predicate R interpreting the teams.*

$\Sigma^1_1(R\downarrow)\ (\neq \emptyset)$ **D**

"Classical" implication

Classical implication in classical **FO**:

$\phi \supset \psi =_{\text{df}} \neg\phi \vee \psi$

Similarly in **D**, we can define:

$\phi \supset \psi =_{\text{df}} \neg\phi \otimes \psi$

$\mathbf{D}^{\supset} = \mathbf{D}$

"Classical" implication

Another possibility:

In **D**, we can define:
$M \models_X \phi \sqsupset \psi$ iff $M \not\models_X \phi$ or $M \models_X \psi$

Consider a new disjunction $\varovee$, called classical disjunction:
$M \models_X \phi \varovee \psi$ iff $M \models_X \phi$ or $M \models_X \psi$
Consider also classical negation $\sim$, defined by:
$M \models_X \sim \phi$ iff $M \not\models_X \phi$

Then
$\phi \sqsupset \psi \equiv (\sim \phi) \varovee \psi$

$\mathbf{D}^{\sqsupset} = \mathbf{D}^{\sim}$

$\mathbf{D}^{\sim}$ =Team logic

$\mathbf{D}^{\sim}$ =Team logic [Väänänen 2007]

Theorem (Väänänen)

On the level of sentences, team logic is equivalent to **SO**.

Theorem (Kontinen, Nurmi)

Restricted to nonempty teams, open formulas of team logic are equivalent to **SO** *sentences with a new predicate R interpreting the teams.*

[Abramsky, Väänänen]
We can define two implications which satisfy the following:

$$\phi \wedge \psi \models \chi \iff \phi \models \psi \to \chi$$

$$\phi \otimes \psi \models \chi \iff \phi \models \psi \multimap \chi$$

Intuitionistic Implication:
$M \models_X \phi \to \psi$ iff for all $Y \subseteq X$, if $M \models_Y \phi$ then $M \models_Y \psi$

Linear Implication:
$M \models_X \phi \multimap \psi$ iff for all Y, if $M \models_Y \phi$ then $M \models_{X \cup Y} \psi$

Both implications preserve downwards closure.
Hence, $\mathbf{D}^{[\to, \multimap]} \neq$ Team logic.

$\to$ preserves empty team property, while $\multimap$ does not.

[Abramsky, Väänänen]
We can define two implications which satisfy the following:

$$\phi \wedge \psi \models \chi \Longleftrightarrow \phi \models \psi \rightarrow \chi$$

$$\phi \otimes \psi \models \chi \Longleftrightarrow \phi \models \psi \multimap \chi$$

Intuitionistic Implication:
$M \models_X \phi \rightarrow \psi$ iff for all $Y \subseteq X$, if $M \models_Y \phi$ then $M \models_Y \psi$

Linear Implication:
$M \models_X \phi \multimap \psi$ iff for all Y, if $M \models_Y \phi$ then $M \models_{X \cup Y} \psi$

Both implications preserve downwards closure.
Hence, $\mathbf{D}^{[\rightarrow, \multimap]} \neq$ Team logic.

$\rightarrow$ preserves empty team property, while $\multimap$ does not.

[Abramsky, Väänänen]
We can define two implications which satisfy the following:

$$\phi \wedge \psi \models \chi \iff \phi \models \psi \to \chi$$

$$\phi \otimes \psi \models \chi \iff \phi \models \psi \multimap \chi$$

Intuitionistic Implication:
$M \models_X \phi \to \psi$ iff for all $Y \subseteq X$, if $M \models_Y \phi$ then $M \models_Y \psi$

Linear Implication:
$M \models_X \phi \multimap \psi$ iff for all Y, if $M \models_Y \phi$ then $M \models_{X \cup Y} \psi$

Both implications preserve downwards closure.
Hence, $\mathbf{D}^{[\to, \multimap]} \neq$Team logic.

$\to$ preserves empty team property, while $\multimap$ does not.

[Abramsky, Väänänen]
We can define two implications which satisfy the following:

$$\phi \wedge \psi \models \chi \iff \phi \models \psi \to \chi$$

$$\phi \otimes \psi \models \chi \iff \phi \models \psi \multimap \chi$$

Intuitionistic Implication:
$M \models_X \phi \to \psi$ iff for all $Y \subseteq X$, if $M \models_Y \phi$ then $M \models_Y \psi$

Linear Implication:
$M \models_X \phi \multimap \psi$ iff for all Y, if $M \models_Y \phi$ then $M \models_{X \cup Y} \psi$

Both implications preserve downwards closure.
Hence, $\mathbf{D}^{[\to,\multimap]} \neq$ Team logic.

$\to$ preserves empty team property, while $\multimap$ does not.

[Abramsky, Väänänen]
We can define two implications which satisfy the following:

$$\phi \wedge \psi \models \chi \iff \phi \models \psi \to \chi$$
$$\phi \otimes \psi \models \chi \iff \phi \models \psi \multimap \chi$$

Intuitionistic Implication:
$M \models_X \phi \to \psi$ iff for all $Y \subseteq X$, if $M \models_Y \phi$ then $M \models_Y \psi$

Linear Implication:
$M \models_X \phi \multimap \psi$ iff for all Y, if $M \models_Y \phi$ then $M \models_{X \cup Y} \psi$

Both implications preserve downwards closure.
Hence, $\mathbf{D}^{[\to, \multimap]} \neq$Team logic.

$\to$ preserves empty team property, while $\multimap$ does not.

Break dependence atom into pieces [Abramsky, Väänänen]

$$=\!(x_1, \cdots, x_n, y) \equiv \big(=\!(x_1) \wedge \cdots \wedge =\!(x_n) \big) \rightarrow =\!(y)$$

Break dependence atom into pieces [Abramsky, Väänänen]

$$=(x_1, \cdots, x_n, y) \equiv (\,=(x_1) \wedge \cdots \wedge =(x_n)\,) \rightarrow =(y)$$

$M = \{a, b, c, d, e\}, \quad X = \{s_0, s_1, s_2, s_3, s_4, s_5\}$
$M \models_X =(x_1, x_2, y)$ iff $M \models_X (\,=(x_1) \wedge =(x_2)\,) \rightarrow =(y)$

	x_1	x_2	y
s_0	a	b	c
s_1	a	b	c
s_2	b	d	e
s_3	b	d	e
s_4	a	b	c
s_5	a	b	c

Break dependence atom into pieces [Abramsky, Väänänen]

$$=(x_1, \cdots, x_n, y) \equiv \left(=(x_1) \wedge \cdots \wedge =(x_n) \right) \rightarrow =(y)$$

$M = \{a, b, c, d, e\}, \quad X = \{s_0, s_1, s_2, s_3, s_4, s_5\}$
$M \models_X =(x_1, x_2, y)$ iff $M \models_X \left(=(x_1) \wedge =(x_2) \right) \rightarrow =(y)$

	x_1	x_2	y
s_0	a	b	c
s_1	a	b	c
s_2	b	d	e
s_3	b	d	e
s_4	a	b	c
s_5	a	b	c

Break dependence atom into pieces [Abramsky, Väänänen]

$$=(x_1, \cdots, x_n, y) \equiv \big(=(x_1) \wedge \cdots \wedge =(x_n) \big) \to =(y)$$

$M = \{a, b, c, d, e\}, \quad X = \{s_0, s_1, s_2, s_3, s_4, s_5\}$
$M \models_X =(x_1, x_2, y)$ iff $M \models_X \big(=(x_1) \wedge =(x_2) \big) \to =(y)$

	x_1	x_2	y
s_0	a	b	c
s_1	a	b	c
s_2	b	d	e
s_3	b	d	e
s_4	a	b	c
s_5	a	b	c

Break dependence atom into pieces [Abramsky, Väänänen]

$$=\!(x_1,\cdots,x_n,y) \equiv \big(\,=\!(x_1)\wedge\cdots\wedge =\!(x_n)\,\big)\to =\!(y)$$

$M=\{a,b,c,d,e\}$, $\quad X=\{s_0,s_1,s_2,s_3,s_4,s_5\}$

$M\models_X =\!(x_1,x_2,y)$ iff $M\models_X \big(\,=\!(x_1)\wedge =\!(x_2)\,\big)\to =\!(y)$

	x_1	x_2	y
s_0	a	b	c
s_1	a	b	c
s_2	b	d	e
s_3	b	d	e
s_4	a	b	c
s_5	a	b	c

Armstrong's Axioms v.s. Heyting's axioms of Intuitionistic Logic [Abramsky, Väänänen]

Armstrong's Axioms	Heyting's Axioms of Intuitionistic Logic
$=\!(x,x)$	$=\!(x) \to =\!(x)$
If $=\!(x,y,z)$, then $=\!(y,x,z)$	If $=\!(x) \wedge =\!(y) \to =\!(z)$, then $=\!(y) \wedge =\!(x) \to =\!(z)$
If $=\!(x,x,y)$, then $=\!(x,y)$	If $=\!(x) \wedge =\!(x) \to =\!(y)$, then $=\!(x) \to =\!(y)$
If $=\!(x,z)$, then $=\!(x,y,z)$	If $=\!(x) \to =\!(z)$, then $=\!(x) \wedge =\!(y) \to =\!(z)$
If $=\!(x,y)$ and $=\!(y,z)$, then $=\!(x,z)$	If $=\!(x) \to =\!(y)$ and $=\!(y) \to =\!(z)$, then $=\!(x) \to =\!(z)$

Expressive power of $\mathbf{D}^{\rightarrow}$ sentences

Theorem

$\mathbf{D}^{\rightarrow}$ *is equivalent to* **SO**, *on the level of sentences.*

Proof. For example, the **SO** sentence

$$\phi := \forall f \exists g \forall x (fx \neq gx),$$

is equivalent to the $\mathbf{D}^{\rightarrow}$ sentence

$$\phi^* := \forall x \forall u (=(x,u) \rightarrow \exists v (u \neq v)),$$

in the sense that for any model M,

$$M \models \phi \iff M \models_{\emptyset} \phi^*.$$

□

Expressive power of $\mathbf{D}^{\rightarrow}$ sentences

Theorem

$\mathbf{D}^{\rightarrow}$ *is equivalent to* **SO***, on the level of sentences.*

Proof. For example, the **SO** sentence

$$\phi := \forall f \exists g \forall x (fx \neq gx),$$

is equivalent to the $\mathbf{D}^{\rightarrow}$ sentence

$$\phi^* := \forall x \forall u \big(=\!(x, u) \rightarrow \exists v (u \neq v)\big),$$

in the sense that for any model M,

$$M \models \phi \iff M \models_{\emptyset} \phi^*.$$

□

Expressive power of $\mathbf{D}^{\rightarrow}$ sentences

Theorem

$\mathbf{D}^{\rightarrow}$ *is equivalent to* **SO**, *on the level of sentences.*

Proof. For example, the **SO** sentence

$$\phi := \forall f \exists g \forall x (fx \neq gx),$$

is equivalent to the $\mathbf{D}^{\rightarrow}$ sentence

$$\phi^* := \forall x \forall u \big((=\!(x) \rightarrow =\!(u)) \rightarrow \exists v (u \neq v) \big),$$

in the sense that for any model M,

$$M \models \phi \iff M \models_{\emptyset} \phi^*. \quad \square$$

In fact, "constancy $\mathbf{D}^{\rightarrow}$ = **SO**",
although "constancy **D** = **FO**", on sentence level. [Galliani]

Expressive power of $\mathbf{D}^{\rightarrow}$ sentences

Theorem

$\mathbf{D}^{\rightarrow}$ *is equivalent to* **SO***, on the level of sentences.*

Proof. For example, the **SO** sentence

$$\phi := \forall f \exists g \forall x (fx \neq gx),$$

is equivalent to the $\mathbf{D}^{\rightarrow}$ sentence

$$\phi^* := \forall x \forall u \big((=\!(x) \rightarrow =\!(u)) \rightarrow \exists v (u \neq v) \big),$$

in the sense that for any model M,

$$M \models \phi \iff M \models_{\emptyset} \phi^*. \quad \square$$

In fact, "constancy $\mathbf{D}^{\rightarrow}$ = **SO**",
although "constancy $\mathbf{D}$ = **FO**", on sentence level. [Galliani]

Expressive power of $\mathbf{D}^{\rightarrow}$, $\mathbf{D}^{[\rightarrow,\multimap]}$ sentences

On sentences level

Theorem

$\mathbf{D}^{\rightarrow}$ *is equivalent to* **SO**.

Corollary

$\mathbf{D}^{[\rightarrow,\multimap]}$ *is equivalent to* **SO**.

SO	$\mathbf{D}^{\rightarrow}$, Constancy $\mathbf{D}^{\rightarrow}$, $\mathbf{D}^{[\rightarrow,\multimap]}$
Σ_1^1	**D**
FO	**FO**, $\mathbf{FO}^{\rightarrow}$, Constancy **D**

Definition

Let R be a k-ary relation symbol and $\phi(R)$ a second order $L(R)$ sentence. We say that $\phi(R)$ is *downwards monotone* with respect to R if for any $L(R)$ model (M, Q) and $Q' \subseteq Q$,

$$(M, Q) \models \phi(R) \implies (M, Q') \models \phi(R).$$

Expressive power of $\mathbf{D}^{[\to,\multimap]}$ open formulas

Any team X of M with $dom(X) = \{x_1, \ldots, x_n\}$ corresponds to a relation on M:

$$rel(X) = \{(s(x_1), \ldots, s(x_n)) \mid s \in X\}$$

Theorem

For any $\mathbf{D}^{[\to,\multimap]}$ L-formula $\phi(\bar{x})$, there exists a **SO** *$L(R)$-sentence $\psi(R)$ downwards monotone w.r.t. R such that for any L-model M, any team X,*

$$M \models_X \phi(\bar{x}) \iff (M, rel(X)) \models \psi(R).$$

Expressive power of $\mathbf{D}^{[\to,\multimap]}$ open formulas

Any team X of M with $dom(X) = \{x_1, \dots, x_n\}$ corresponds to a relation on M:

$$rel(X) = \{(s(x_1), \dots, s(x_n)) \mid s \in X\}$$

Theorem

For any $\mathbf{D}^{[\to,\multimap]}$ *L-formula* $\phi(\bar{x})$*, there exists a* **SO** $L(R)$*-sentence* $\psi(R)$ *downwards monotone w.r.t.* R *such that for any L-model* M*, any team* X*,*

$$M \models_X \phi(\bar{x}) \iff (M, rel(X)) \models \psi(R).$$

Expressive power of $\mathbf{D}^{[\to,\multimap]}$ open formulas

Theorem

For any **SO** $L(R)$*-sentence* $\phi(R)$ *downwards monotone w.r.t.* R*, there is a* $\mathbf{D}^{\to}$ L*-formula* $\psi(\bar{x})$ *such that for any* L*-model* M*, any nonempty team* X

$$(M, rel(X)) \models \phi(R) \iff M \models_X \psi(\bar{x}).$$

Proposition

For any **SO** $L(R)$*-sentence* $\phi(R)$ *downwards monotone w.r.t.* R*, there is a* $\mathbf{D}^{[\to,\multimap]}$ L*-formula* $\chi(\bar{x})$ *such that for any* L*-model* M*,*

$$(M, rel(\emptyset)) \models \phi(R) \iff M \models_\emptyset \chi(\bar{x}).$$

Expressive power of $\mathbf{D}^{[\rightarrow,\multimap]}$ open formulas

Theorem

For any **SO** *$L(R)$-sentence $\phi(R)$ downwards monotone w.r.t. R, there is a* $\mathbf{D}^{\rightarrow}$ *L-formula $\psi(\bar{x})$ such that for any L-model M, any nonempty team X*

$$(M, rel(X)) \models \phi(R) \iff M \models_X \psi(\bar{x}).$$

Proposition

For any **SO** *$L(R)$-sentence $\phi(R)$ downwards monotone w.r.t. R, there is a* $\mathbf{D}^{[\rightarrow,\multimap]}$ *L-formula $\chi(\bar{x})$ such that for any L-model M,*

$$(M, rel(\emptyset)) \models \phi(R) \iff M \models_{\emptyset} \chi(\bar{x}).$$

Theorem

For any **SO** *$L(R)$-sentence $\phi(R)$ downwards monotone w.r.t. R, there is a* $\mathbf{D}^{[\rightarrow,\multimap]}$ *L-formula $\theta(\bar{x}) := \psi \otimes (\perp \wedge \chi)$ such that for any L-model M, any team X*

$$(M, rel(X)) \models \phi(R) \iff M \models_X \theta(\bar{x}).$$

Expressive power of $\mathbf{D}^{[\to,\multimap]}$ open formulas

On formulas level

Theorem

Restricted to nonempty teams, $\mathbf{D}^{\to}$ characterizes exactly second order downwards monotone properties.

Theorem

$\mathbf{D}^{[\to,\multimap]}$ characterizes exactly second order downwards monotone properties.

$\mathbf{SO}(R\downarrow)$	$\mathbf{D}^{[\to,\multimap]}$
$\mathbf{SO}(R\downarrow)\ (\neq\emptyset)$	$\mathbf{D}^{\to}$
$\Sigma^1_1(R\downarrow)\ (\neq\emptyset)$	$\mathbf{D}$

Independence logic, Inclusion/Exclusion logic

Well-formed formulas of **Ind** (in negation normal form) are given by the following grammar

$$\phi ::= \alpha \mid \bar{x} \perp_{\bar{z}} \bar{y} \mid \phi \wedge \phi \mid \phi \otimes \phi \mid \forall x \phi \mid \exists x \phi$$

Well-formed formulas of **I**/**E**-logic (in negation normal form) are given by the following grammar

$$\phi ::= \alpha \mid \overline{x} \subseteq \overline{y} \mid \overline{x} \mid \overline{y} \mid \phi \wedge \phi \mid \phi \otimes \phi \mid \forall x \phi \mid \exists x \phi$$

- $M \models_X \bar{x} \perp_{\bar{z}} \bar{y}$ iff for all $s, s' \in X$ such that $s(\bar{z}) = s'(\bar{z})$, there exists $s'' \in X$ such that $s''(\bar{z}) = s'(\bar{z}) = s(\bar{z})$, $s''(\bar{x}) = s(\bar{x})$ and $s''(\bar{y}) = s(\bar{y})$.
- $M \models_X \bar{x} \mid \bar{y}$ with $|\bar{x}| = |\bar{y}|$ iff $\forall s, s' \in X$, $s(\bar{x}) \neq s'(\bar{y})$.
- $M \models_X \bar{x} \subseteq \bar{y}$ with $|\bar{x}| = |\bar{y}|$ iff $\forall s \in X$, $\exists s' \in X$ such that $s'(\bar{y}) = s(\bar{x})$.
- (Lax semantics) $M \models_X \exists x \varphi$ iff there is a function $F : X \to \wp(M) \setminus \{\emptyset\}$ such that $M \models_{X[F/x]} \varphi$, where

$$X[F/x] = \{s(a/x) \mid s \in X, a \in F(s)\}.$$

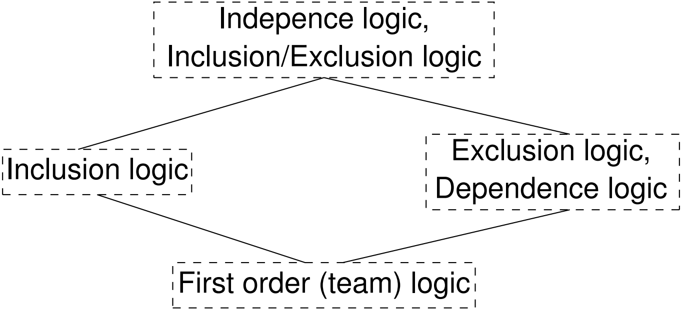
Constancy independence atom

$$M \models_X x \perp_\emptyset x \text{ iff for all } s, s' \in X \; s(x) = s'(x).$$

	$\cdots$	x	$\cdots$
s_0		a	
s_1		a	
s_2		a	
s_3		a	
s_4		a	
s_5		a	

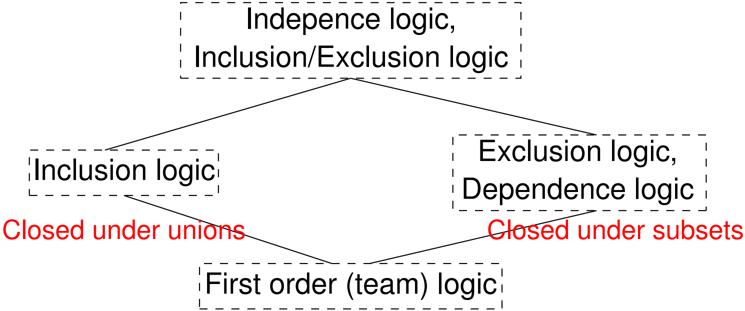
Expressive power of **Ind**, **I,E**

On formulas level [Galliani]



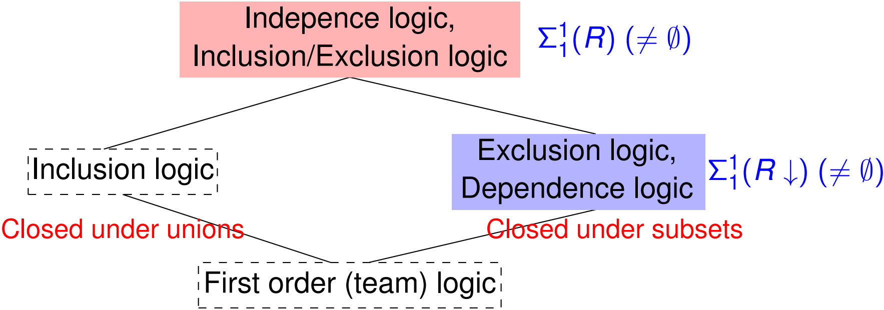
Expressive power of **Ind**, **I**,**E**

On formula level [Galliani]



Expressive power of **Ind**, **I,E**

On formula level [Galliani]



Expressive power of **Ind, I,E**

On sentence level [Väänänen, Grädel, Galliani]

Σ_1^1	**Ind, I/E, E, D**
FO	**FO** (team), Constancy **D**

Intuitionistic Implication and Linear Implication

In **Ind**

$$\phi \wedge \psi \models \chi \not\Leftrightarrow \phi \models \psi \to \chi$$

$$\phi \otimes \psi \models \chi \not\Leftrightarrow \phi \models \psi \multimap \chi$$

Expressive power of $\mathbf{Ind}^{\rightarrow}$, $\mathbf{Ind}^{[\rightarrow,\multimap]}$

For sentences:

Theorem

$\mathbf{Ind}^{\rightarrow}$ and $\mathbf{Ind}^{[\rightarrow,\multimap]}$ are equivalent to **SO** *on the level of sentences*

SO	$\mathbf{D}^{\rightarrow}$, $\mathbf{E}^{\rightarrow}$, $(\mathbf{I/E})^{\rightarrow}$, $\mathbf{Ind}^{\rightarrow}$, $\mathbf{Ind}^{[\rightarrow,\multimap]}$
Σ_1^1	**Ind, I/E, E, D**
FO	**FO** (team)

Expressive power of $\mathbf{Ind}^{[\rightarrow,\multimap]}$

For open formulas:

- One direction:

Theorem

For any $\mathbf{Ind}^{[\rightarrow,\multimap]}$ *L-formula* $\phi(\bar{x})$*, there exists a* **SO** $L(R)$*-sentence* $\psi(R)$ *downwards monotone w.r.t.* R *such that for any L-model* M*, any team* X*,*

$$M \models_X \phi(\bar{x}) \iff (M, rel(X)) \models \psi(R).$$

Expressive power of **Ind**$^{[\to,\multimap]}$

For open formulas:

- The other direction:

Theorem

For any **SO** $L(R)$*-sentence* $\phi(R)$*, there exists a* **Ind**$^{\to}$ L*-formula* $\psi(\bar{x})$ *such that for any* L*-model* M*, any nonempty team* X

$$(M, rel(X)) \models \phi(R) \iff M \models_X \psi(\bar{x}).$$

Proposition

For any **SO** $L(R)$*-sentence* $\phi(R)$*, there exists a* **Ind**$^{[\to,\multimap]}$ L*-formula* $\chi(\bar{x})$ *such that for any* L*-model* M*,*

$$(M, rel(\emptyset)) \models \phi(R) \iff M \models_\emptyset \chi(\bar{x}).$$

Expressive power of $\mathbf{Ind}^{[\to, \multimap]}$

For open formulas:

- The other direction:

Theorem

For any **SO** $L(R)$*-sentence* $\phi(R)$*, there exists a* $\mathbf{Ind}^{\to}$ L*-formula* $\psi(\bar{x})$ *such that for any* L*-model* M*, any nonempty team* X

$$(M, rel(X)) \models \phi(R) \iff M \models_X \psi(\bar{x}).$$

Proposition

For any **SO** $L(R)$*-sentence* $\phi(R)$*, there exists a* $\mathbf{Ind}^{[\to, \multimap]}$ L*-formula* $\chi(\bar{x})$ *such that for any* L*-model* M*,*

$$(M, rel(\emptyset)) \models \phi(R) \iff M \models_\emptyset \chi(\bar{x}).$$

Expressive power of $\mathbf{Ind}^{\rightarrow}$

On formulas level

Theorem

Restricted to nonempty teams, $\mathbf{Ind}^{\rightarrow}$ *characterizes exactly second order properties.*

$\mathbf{SO}(R)\ (\neq \emptyset)$	$\mathbf{Ind}^{\rightarrow}$
$\mathbf{SO}(R\downarrow)$	$\mathbf{D}^{[\rightarrow, \multimap]}$
$\mathbf{SO}(R\downarrow)\ (\neq \emptyset)$	$\mathbf{D}^{\rightarrow}$
$\Sigma_1^1(R\downarrow)\ (\neq \emptyset)$	$\mathbf{D}$

Expressive power of $\mathbf{Ind}^{[\rightarrow,\multimap]}$

$\mathbf{Ind}^{[\rightarrow,\multimap]} \neq$ Team Logic

Theorem (Kontinen, Nurmi)

For every formula ϕ of team logic one of the following holds:

- $M \models_{\emptyset} \phi$ *for all* M
- $M \not\models_{\emptyset} \phi$ *for all* M

In $\mathbf{Ind}^{[\rightarrow,\multimap]}$,

- $M \models_{\emptyset} \top \multimap \exists x \forall y (x = y)$ iff $|M| = 1$

Break independence atom into pieces

$$=\!(x_1, \cdots, x_n, y) \equiv \big(=\!(x_1) \wedge \cdots \wedge =\!(x_n) \big) \to =\!(y)$$

$$=\!(x) \equiv x \perp_{\emptyset} x$$

$$\bar{x} \perp_{\bar{z}} \bar{y} \equiv \big((z_1 \perp_{\emptyset} z_1) \wedge \cdots \wedge (z_n \perp_{\emptyset} z_n)\big) \to (\bar{x} \perp_{\emptyset} \bar{y})?$$

Break independence atom into pieces

$$=\!(x_1, \cdots, x_n, y) \equiv \big(=\!(x_1) \wedge \cdots \wedge =\!(x_n) \big) \to =\!(y)$$

$$=\!(x) \equiv x \perp_{\emptyset} x$$

$$\bar{x} \perp_{\bar{z}} \bar{y} \equiv \big((z_1 \perp_{\emptyset} z_1) \wedge \cdots \wedge (z_n \perp_{\emptyset} z_n)\big) \to (\bar{x} \perp_{\emptyset} \bar{y})?$$

Break independence atom into pieces

$$=\!(x_1, \cdots, x_n, y) \equiv \left(=\!(x_1) \wedge \cdots \wedge =\!(x_n) \right) \to =\!(y)$$

$$=\!(x) \equiv x \perp_{\emptyset} x$$

$$\bar{x} \perp_{\bar{z}} \bar{y} \equiv \big((z_1 \perp_{\emptyset} z_1) \wedge \cdots \wedge (z_n \perp_{\emptyset} z_n)\big) \to (\bar{x} \perp_{\emptyset} \bar{y})?$$

Break independence atom into pieces

$$=\!(x_1, \cdots, x_n, y) \equiv \big(=\!(x_1) \wedge \cdots \wedge =\!(x_n) \big) \to =\!(y)$$

$$=\!(x) \equiv x \perp_{\emptyset} x$$

$$\bar{x} \perp_{\bar{z}} \bar{y} \not\equiv \big((z_1 \perp_{\emptyset} z_1) \wedge \cdots \wedge (z_n \perp_{\emptyset} z_n)\big) \to (\bar{x} \perp_{\emptyset} \bar{y})$$

Maximal implication

Definition (Maximal implication)

$M \models_X \phi \hookrightarrow \psi$ iff for all maximal $Y \subseteq X$ such that $M \models_Y \phi$, it holds that $M \models_Y \psi$.

$\hookrightarrow$ preserves empty team property.

Break independence atom into pieces

$$\bar{x} \perp_{\bar{z}} \bar{y} \equiv ((z_1 \perp_\emptyset z_1) \wedge \cdots \wedge (z_n \perp_\emptyset z_n)) \hookrightarrow (\bar{x} \perp_\emptyset \bar{y})$$

Example: $x \perp_{z_1 z_2} y \equiv ((z_1 \perp_\emptyset z_1) \wedge (z_2 \perp_\emptyset z_2)) \hookrightarrow (x \perp_\emptyset y)$

	z_1	z_2	x	y
s_0	a	b	b	c
s_1	a	b	d	e
s_2	b	c	d	c
s_3	c	d	b	c
s_4	a	b	b	e
s_5	a	b	d	c

Break independence atom into pieces

$$\bar{x} \perp_{\bar{z}} \bar{y} \equiv \big((z_1 \perp_{\emptyset} z_1) \wedge \cdots \wedge (z_n \perp_{\emptyset} z_n)\big) \hookrightarrow (\bar{x} \perp_{\emptyset} \bar{y})$$

Example: $x \perp_{z_1 z_2} y \equiv \big((z_1 \perp_{\emptyset} z_1) \wedge (z_2 \perp_{\emptyset} z_2)\big) \hookrightarrow (x \perp_{\emptyset} y)$

	z_1	z_2	x	y
s_0	a	b	b	c
s_1	a	b	d	e
s_2	b	c	d	c
s_3	c	d	b	c
s_4	a	b	b	e
s_5	a	b	d	c

Break independence atom into pieces

$$\bar{x} \perp_{\bar{z}} \bar{y} \equiv ((z_1 \perp_\emptyset z_1) \wedge \cdots \wedge (z_n \perp_\emptyset z_n)) \hookrightarrow (\bar{x} \perp_\emptyset \bar{y})$$

Example: $x \perp_{z_1 z_2} y \equiv ((z_1 \perp_\emptyset z_1) \wedge (z_2 \perp_\emptyset z_2)) \hookrightarrow (x \perp_\emptyset y)$

	z_1	z_2	x	y
s_0	a	b	b	c
s_1	a	b	d	e
s_2	b	c	d	c
s_3	c	d	b	c
s_4	a	b	b	e
s_5	a	b	d	c

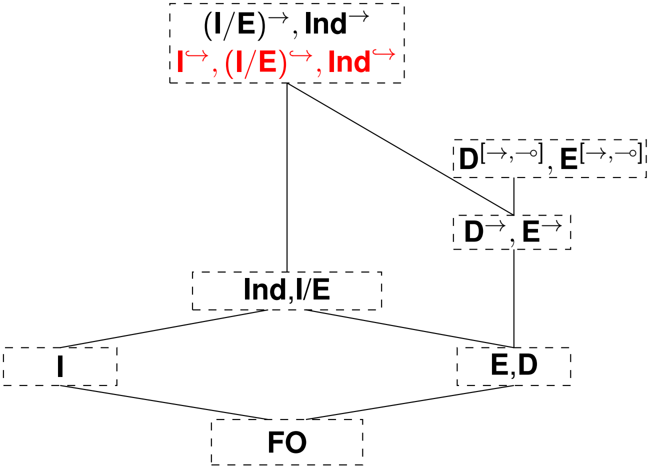
Expressive power of logics

On sentence level

SO	$\mathbf{D}^{\rightarrow}, \mathbf{E}^{\rightarrow}, (\mathbf{I}/\mathbf{E})^{\rightarrow}, \mathbf{Ind}^{\rightarrow}, \mathbf{Ind}^{[\rightarrow, \multimap]}$ $\mathbf{D}^{\hookrightarrow}, \mathbf{E}^{\hookrightarrow}, \mathbf{I}^{\hookrightarrow}, (\mathbf{I}/\mathbf{E})^{\hookrightarrow}, \mathbf{Ind}^{\hookrightarrow}$
Σ^1_1	**Ind, I/E, E, D**
FO	**FO** (team)

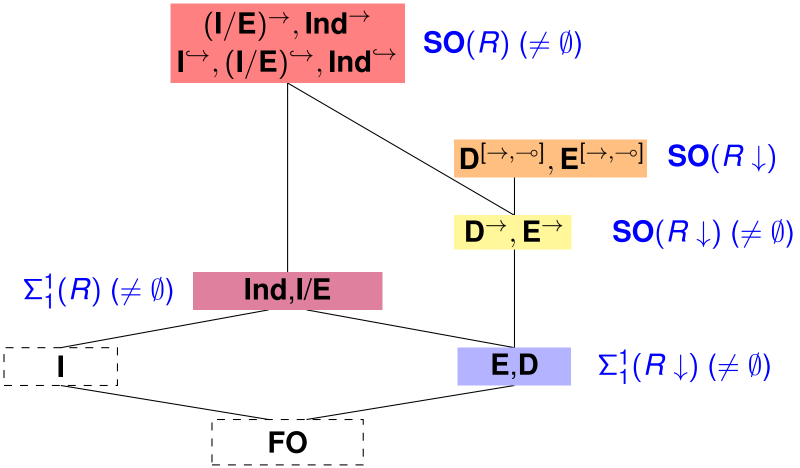
Expressive power of logics

On formula level



Expressive power of logics

On formula level



That's all!

Thank you!